Logistic Model: A population $y(t)$ at time t satisfies

$$\frac{dy}{dt} = ky - ly^2 \qquad (*)$$

for k, l constants and $l > 0$ much smaller than k. This is a separable differential equation since we can rewrite it as

$$\frac{1}{ky - ly^2} \frac{dy}{dt} = 1 \qquad (*)$$

we integrate both sides to solve $(*)$.

$$\int \frac{1}{ky - ly^2} \, dy = \int dt \qquad (**)$$

For the left hand side of (*) we use partial fractions.

$$\frac{1}{ky-ly^2} = \frac{1}{(k-ly)y} = \frac{A}{y} + \frac{B}{k-ly}$$

Let's work out A & B.

$$\frac{1}{(k-ly)y} = \frac{A(k-ly)+By}{(k-ly)y}$$

$$1 = A(k-ly) + By$$

$\underline{\underline{y=0}}$ $\quad 1 = Ak \qquad \boxed{A = \frac{1}{k}}$

$y = \frac{k}{l} \quad 1 = B\frac{k}{l} \qquad \boxed{B = \frac{l}{k}}$

So (**) becomes

$$\frac{1}{k}\int \frac{1}{y}\,dy + \frac{l}{k}\int \frac{1}{k-ly}\,dy = \int dt$$

$$\frac{1}{k}\ln|y| - \frac{l}{k}\frac{1}{l}\ln|k-ly| = t + c$$

$$\frac{1}{k}\ln|y| - \frac{1}{k}\ln|k-ly| = t + c$$

$$\frac{1}{k}\left(\ln|y| - \ln|k-ly|\right) = t + c$$

$$\frac{1}{k}\ln\left|\frac{y}{k-ly}\right| = t + c$$

$$\ln\left|\frac{y}{k-ly}\right| = kt + c$$

$$\frac{y}{k-ly} = e^{kt+c} = e^{kt}\overset{A}{\boxed{e^{c}}}$$

$$y = Ae^{kt}(k - ly)$$

$$y + Ae^{kt}ly = Ae^{kt}k$$

$$y(1 + Ae^{kt}l) = Ake^{kt}$$

$$\boxed{y = \frac{Ake^{kt}}{1 + Ale^{kt}}}$$

As $t \to \infty$ we have

$$y \to \frac{Ake^{kt}}{Ale^{kt}} \to \frac{k}{l}.$$

Thus the world population, according to the Logistic model, will tend to a constant size $\frac{k}{l}$.

- Geographers have estimated $k = 0.029$

- when $y = (3.06)10^9$ the population was increasing at 2% per year.

$$\frac{dy}{dt} = ky - ly^2$$

$$\frac{1}{y}\frac{dy}{dt} = (k - ly)$$

$$\frac{10^{-9}}{3.06} \times 0.02 = k - l\, 3.09 \times 10^9 ,$$

Since we know k, we can find

$l = 2.941 \times 10^{-12}$

So the world population should tend to

$$\frac{k}{l} = \frac{0.029}{2.941} 10^{12} = 9.86 \text{ billion people.}$$