What will the world's population be in 100 years time?

yesterday: assumed that the population $y(t)$ satisfies the Malthusian Law

$$\frac{dy}{dt} = ky \qquad (*)$$

Solutions to $(*)$ have the form

$$y = Ae^{kt}$$

for some constant A.

From real data we estimated the constants A and k. The resulting function

$$y = 3.06\, e^{0.0198t} \text{ billion}$$

agrees well with data for the period 1700–1960. But the

function y is not so good for future / current periods.

The Malthusian Law ignores competition for resources.

The <u>Logistic Model</u> of population growth is

$$\frac{dy}{dt} = ky - \ell y^2 \qquad (**)$$

where k, ℓ are constants, and ℓ is small in comparison to k. For small y the term ℓy^2 is insignificant and (**) will be very similar to (*). But for larger y the term ℓy^2 will become important, and will slow down the rate of growth.

We need to:

1) Find the general solutions to (**).

2) Then use real data to determine k, l and other constanst in the solutions.

But how do we solve (**)?

Separable Differential Equations

A differential equation is **separable** if it has the form

$$f(y)\,\frac{dy}{dt} = g(t)$$

for some functions f and g.

Example

$$y^2\,\frac{dy}{dt} = t^2$$

is a separable differential eqⁿ.

Example

$$\frac{dy}{dt} = ky - ly^2$$

is separable because it can be rewritten as

$$\frac{1}{ky - ly^2}\,\frac{dy}{dt} = 1$$

How do we solve a separable diff. eq^n.

$$f(y)\,\frac{dy}{dt} = g(t).$$

Such an equation can be rewritten as

$$\frac{d}{dt}F(y) = f(y)\,\frac{dy}{dt} = g(t)$$

where $F(y)$ is any anti-derivative of $f(y)$ (i.e. $\frac{d}{dy}F(y) = f(y)$). Now integrating with respect to t, we find

$$\boxed{F(y) = \int g(t)\,dt + C}$$

Example Solve

$$e^y \frac{dy}{dt} - t - t^3 = 0$$

Sol^n

$$e^y \frac{dy}{dt} = t^3 + t$$

$$\int e^y \, dy = \int t^3 + t \, dt + c$$

$$e^y = \frac{t^4}{4} + \frac{t^2}{2} + c$$

$$\ln(e^y) = \ln\left(\frac{t^4}{4} + \frac{t^2}{2} + c\right)$$

$$y = \ln\left(\frac{t^4}{4} + \frac{t^2}{2} + c\right).$$

Example Solve

$$\frac{dy}{dt} = 1 + y^2, \qquad y(0) = 0$$

Sol^n

$$\frac{1}{1+y^2}\frac{dy}{dt} = 1$$

$$\int \frac{1}{1+y^2}\,dy = \int dt + C$$

$$y = \tan\theta$$

$$dy = \sec^2\theta\,d\theta$$

$$\int \frac{1}{1+\tan^2\theta}\sec^2\theta\,d\theta = \int dt + C$$

$$= \int d\theta = \int dt + C$$

$\theta = t + C$

now

$\underline{\underline{t=0}}, y=0 \Rightarrow 0 = \tan\theta$

$\Rightarrow \underline{\underline{\theta = 0}}.$

Hence, for $t=0$

$0 = 0 + C$

and $C = 0.$

Thus

$\theta = t$

and

$$\boxed{y = \tan(t)}$$

is our solution.

Example Solve

$$y^2 \frac{dy}{dt} = t^2 .$$

Sol^n

$$\int y^2\, dy = \int t^2\, dt + c$$

$$\frac{1}{3}y^3 = \frac{1}{3}t^3 + c$$

$$y^3 = t^3 + c$$

$$y = (t^3 + c)^{\frac{1}{3}}$$