Differential Equations

An equation involving a derivative is a differential equation.

e.g.

$$\frac{dy}{dt} = ky \qquad (*)$$

where k is some constant number and y is a function of t, $y = y(t)$.

1) Are there any solutions to $(*)$?

Consider for instance

$$y = e^{kt}.$$

Then $\frac{dy}{dt} = e^{kt}\frac{d}{dt}(kt) = ke^{kt} = ky$

So $y = e^{kt}$ is one solution to $(*)$.

Also $y = 5e^{kt}$ satisfies

$$\underline{\underline{\frac{dy}{dt}}} = 5ke^{kt} = k(5e^{kt}) = \underline{\underline{ky}}$$

$$\boxed{\begin{array}{l}\text{We see that} \\ \qquad y = Ae^{kt} \\ \text{is a solution to } (*) \text{ for} \\ \text{any constant } A.\end{array}}$$

These are the only solutions to $(*)$. [We'll prove this later.]

Problem A cup of coffee in a room at 20°C cools from 80°C to 50°C in five minutes. How long will it take to cool to 40°C?

Soln

Newton: A hot object cools at a rate proportional to the excess of its temperature above room temperature.

Let's introduce

$y(t) =$ temperature of coffee at time t.

$y(0) = 80$

$y(5) = 50$

Question: For what value of t is $y(t) = 40$?

Newton:

$$\boxed{\frac{dy}{dt} = k(y-20)}$$

Consider $z = y - 20$

$z(0) = 60$

$z(5) = 30$

$$\underline{\underline{\frac{dz}{dt}}} = \frac{d}{dt}(y-20) = \frac{dy}{dt} = k(y-20) = \underline{\underline{kz}}$$

Hence $\frac{dz}{dt} = kz$

The solutions to this equation are

$$\boxed{z = Ae^{kt}}$$

where A, k are constants.

Our question: for what value of t does $z(t) = 20$.

$\underline{t = 0}$

$$60 = z(0) = Ae^{k0} = Ae^{0} = A$$

so

$$\boxed{z = 60e^{kt}}$$

$\underline{t = 5}$

$$30 = z(5) = 60e^{5k}$$

or $\boxed{e^{5k} = \frac{1}{2}}$

For what value of t do we have

$$z(t) = 20 \ ?$$

$$20 = z(t) = 60e^{kt}$$

$$\frac{1}{3} = e^{kt}$$

$\boxed{\ln(x^y) = y\ln(x)}$

$$\frac{1}{3} = \left(e^{5k}\right)^{\frac{t}{5}}$$

$$\frac{1}{3} = \left(\frac{1}{2}\right)^{\frac{t}{5}}$$

$$\ln\left(\frac{1}{3}\right) = \ln\left(\left(\frac{1}{2}\right)^{\frac{t}{5}}\right)$$

$$\ln\left(\frac{1}{3}\right) = \frac{t}{5}\ln\left(\frac{1}{2}\right)$$

$$\frac{t}{5} = \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)}$$

Answer

$$t = \frac{5\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)}$$