Recall we have

$$\frac{p(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} =$$

$$\frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_n}{(x-a_n)}$$

where $\deg(p(x)) < n$, and $a_1, a_2, \ldots, a_n$ are distinct numbers.

Comment 3

If the a_i are not all distinct then the following method works

Problem Find the indefinite integral

$$I = \int \frac{1}{x^3 + x^2 - x - 1}\, dx.$$

Sol

Solⁿ Let's factorise

$$f(x) = x^3 + x^2 - x - 1$$

$f(1) = 1 + 1 - 1 - 1 = 0$, so $(x-1)$ divides $f(x)$

$f(-1) = -1 + 1 + 1 - 1 = 0$, so $(x+1)$ divides $f(x)$

$(x-1)(x+1) = x^2 - 1$ divides $f(x)$

$$\begin{array}{r|l} & x + 1 \\ \hline x^2 - 1 & x^3 + x^2 - x - 1 \\ & \underline{x^3 \quad\;\; - x} \\ & \quad x^2 \quad\;\; - 1 \\ & \quad \underline{x^2 \quad\;\; - 1} \\ & \qquad\qquad 0 \end{array}$$

So $(x-1)(x+1)^2 = x^3 + x^2 - x - 1$.

$$\frac{1}{x^3 + x^2 - x - 1} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$\frac{1}{(x-1)(x+1)^2} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

$$1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$\underline{\underline{x=1}}$

$1 = 4A$ $\qquad \boxed{A = \frac{1}{4}}$

$\underline{\underline{x=-1}}$

$1 = -2C$ $\qquad \boxed{C = -\frac{1}{2}}$

$\underline{\underline{x=3}}$

$1 = \frac{1}{4}16 + 8B - \frac{1}{2}\cdot 2$

$1 = 4 + 8B - 1$

$1 - 4 + 1 = 8B$

$-2 = 8B$ $\qquad \boxed{B = -\frac{1}{4}}$

So

$$I = \int \frac{1}{x^3 + x^2 - x - 1}\,dx$$

$$= \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{(x+1)^2}$$

$$= \frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x+1) - \frac{1}{2}\int \frac{du}{u^2}$$

Let $u = x+1$
$du = dx$

$$= \frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x+1) + \frac{1}{2u} + C$$

$$= \frac{1}{4}\ln(x-1) - \frac{1}{4}\ln(x+1) + \frac{1}{2(x+1)} + C.$$

Let's skip comments 4 & 5

Random Example

Find $I = \int \frac{1}{x^2+4}\,dx$

Sol^n

Let $x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$$I = \int \frac{2\sec^2\theta}{4\tan^2\theta + 4}\, d\theta$$

$$= \frac{2}{4} \int \frac{\sec^2\theta}{\tan^2\theta + 1}\, d\theta$$

$$= \frac{1}{2} \int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{2} \int d\theta$$

$$= \frac{1}{2}\theta + C$$

where $\frac{x}{2} = \tan\theta$.