Technique 6: partial fractions

$$\frac{2}{3} + \frac{4}{5} = \frac{2.5 + 4.3}{3.5} = \frac{22}{15}$$

$$\frac{2}{(x-1)} - \frac{1}{(x+2)} = \frac{2(x+2) - (x-1)}{(x-1)(x+2)} = \frac{x+5}{(x-1)(x+2)}$$

$$= \frac{x+5}{x^2+x-2}$$

$$\frac{2}{x-1} + \frac{x+3}{(x-1)^2} = \frac{2(x-1) + (x+3)}{(x-1)^2} = \frac{3x+1}{(x-1)^2}$$

partial fractions

fractions

Problem Find $I = \int \frac{x+5}{x^2+x-2}\, dx$

Sol

$I = 2\int \frac{1}{(x-1)} dx - \int \frac{1}{(x+2)} dx$

$= 2 \ln(x-1) - \ln(x+2) + C.$

<u>But</u>: Given a fraction, how do we express it as a sum of partial fractions?

Answer involves five comments.

<u>Comment 1</u> Using long division if necessary, any fraction of polynomials can be expressed in the form

$$p(x) + \frac{r(x)}{q(x)}$$

where $p(x)$, $q(x)$, $r(x)$ are polynomials with

$$\text{degree}(r(x)) < \text{degree}(q(x))$$

Problem Find $I = \int \frac{x^3+x}{x-1}\,dx$

Sol^n

$$\begin{array}{r|l} & x^2+x \\ \hline x-1 & x^3+x \\ & x^3-x^2 \\ \hline & \quad x^2+x \\ & \quad x^2-x \\ \hline & \qquad 2x \end{array}$$

so $\frac{x^3+x}{x-1} = x^2+x + \frac{2x}{x^2+x}$

$$I = \int x^2+x \,dx + \int \frac{2x}{x^2+x}\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + \ln(x^2+x) + C$$

Comment 2 If a fraction is such that degree of top < degree of bottom, and if the bottom is a product of distinct linear factors, then the following method will work.

Problem Find $I = \int \frac{8x+1}{2x^2-x-1}\,dx$

Sol^n

$$\frac{8x+1}{2x^2-x-1} = \frac{8x+1}{(2x+1)(x-1)} = \frac{A}{(2x+1)} + \frac{B}{(x-1)}$$

for some A, B.

so

$$\frac{8x+1}{(2x+1)(x-1)} = \frac{A(x-1)+B(2x+1)}{(2x+1)(x-1)}$$

So

$$8x+1 = A(x-1) + B(2x+1)$$

$x=1$: $8+1 = A.0 + B(2+1)$

$$9 = 3B$$

$$\boxed{B=3}$$

$x=-\frac{1}{2}$: $-4+1 = A\left(-\frac{3}{2}\right) + B.0$

$$-3 = -\frac{3}{2}A$$

$$\boxed{A=2}$$

So

$$\frac{8x+1}{2x^2-x-1} = \frac{2}{(2x+1)} + \frac{3}{(x-1)}$$

$$I = \int \frac{8x+1}{2x^2-x-1}\,dx = \int \frac{2}{2x+1}\,dx + 3\int \frac{1}{x-1}\,dx$$

$$= \ln(2x+1) + 3\ln(x-1) + C.$$