

Technique 5: Trigonometric Substitutions

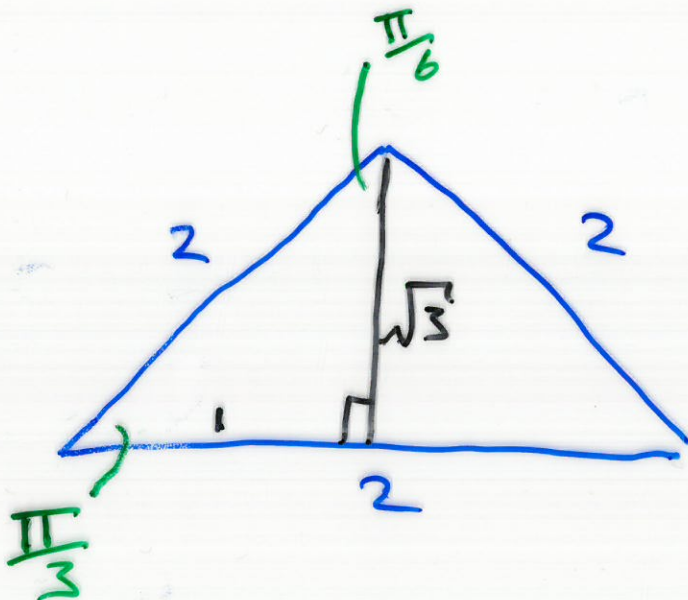
Problem Evaluate

$$I = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\sqrt{1-x^2}}{x^2} dx$$

Soln Let $x = \sin \theta$
 $dx = \cos \theta d\theta$

When $x = \frac{1}{2}$ we have $\theta = \frac{\pi}{6}$

When $x = \frac{\sqrt{3}}{2}$ we have $\theta = \frac{\pi}{3}$



$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{1 - \sin^2 \theta}}{\sin^2 \theta} \cos \theta \, d\theta$$

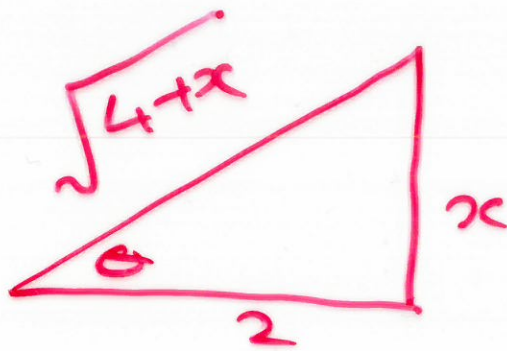
$$= \int_{\pi/6}^{\pi/3} \frac{\cos^2 \theta}{\sin^2 \theta} \, d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1 - \sin^2 \theta}{\sin^2 \theta} \, d\theta$$

$$= \int_{\pi/6}^{\pi/3} \frac{1}{\sin^2 \theta} - 1 \, d\theta$$

$$= \int_{\pi/6}^{\pi/3} \operatorname{cosec}^2 \theta - 1 \, d\theta$$

$$= -\frac{1}{4} \sin \theta + C$$



$$I = -\frac{1}{4} \frac{1}{\left(\frac{x}{\sqrt{4+x}} \right)} + C$$

$$I = -\frac{\sqrt{4+x}}{4x} + C.$$

Table of trig. substitutions

Expression	substitution	identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \cos^2 \theta = \sin^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$

Problem Find

$$I = \int \frac{1}{x^2 \sqrt{x^2 + 4}} dx$$

Solⁿ Let $x = 2 \tan \theta$

$$dx = 2 \sec^2 \theta d\theta$$

$$I = \int \frac{2 \sec^2 \theta \, d\theta}{4 \tan^2 \theta \sqrt{4 \tan^2 \theta + 4}}$$

$$= \frac{\cancel{2}}{4} \int \frac{\sec^2 \theta \, d\theta}{\tan^2 \theta \cancel{2} \sqrt{\tan^2 \theta + 1}}$$

$$= \frac{1}{4} \int \frac{\sec \cancel{\theta} \, d\theta}{\tan^2 \theta \cancel{\sec \theta}}$$

$$= \frac{1}{4} \int \frac{\sec \theta}{\tan^2 \theta} \, d\theta$$

$$= \frac{1}{4} \int \frac{\cos^2 \theta}{\cancel{\cos \theta} \sin^2 \theta} \, d\theta$$

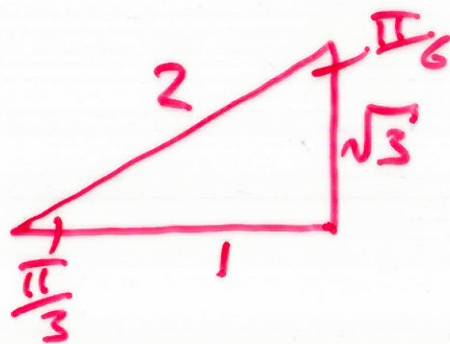
$$= \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} \, d\theta$$

$$= \frac{1}{4} \int \cos \theta \cdot \sin^{-2} \theta \, d\theta$$

$$= -\cot \theta - \theta \Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= -\cot \frac{\pi}{3} - \frac{\pi}{2} - \left(-\cot \frac{\pi}{6} - \frac{\pi}{6} \right)$$

$$= -\cot \frac{\pi}{3} + \cot \frac{\pi}{6} - \frac{\pi}{6}$$



$$\cot(\theta) = \frac{1}{\tan \theta} = \frac{a}{o}$$

$$= -\frac{1}{\sqrt{3}} + \sqrt{3} - \frac{\pi}{6} .$$