Problem A stone is dropped from a high cliff. The acceleration due to gravity is $9.8\ m/sec^2$. How fast is the stone travelling 5 seconds after being dropped?

Soln The acceleration $a(t)$ at time t is just the rate of change of velocity with respect to time. So, at time T, the velocity is

$$v(T) = \int_0^T a(t)\,dt \qquad (*)$$

$$v(5) = \int_0^5 9.8\,dt$$

$$v(5) = 9.8t \Big|_0^5$$

$$= 9.8(5) - 0$$

$$= 49 \text{ m/s}.$$

Problem (Continued).
The cliff is 200m high. How long does the stone take to reach the bottom?

Soln
From (*) above the speed at time T is

$$v(T) = \int_0^T a(t)\,dt = \int_0^T 9.8\,dt$$

$$= 9.8t \Big|_0^T = 9.8T.$$

So, at time T, the stone will have travelled a

$$\text{distance} = \int_0^T v(t)\,dt = \int_0^T 9.8t\,dt$$

$$= \frac{9.8t^2}{2}\Big|_0^T$$

$$= 4.9T^2 - 0$$

$$= 4.9T^2.$$

The time T taken for the stone to reach the ground is found by solving

$$200 = 4.9T^2$$

$$T = \sqrt{\frac{200}{4.9}} = 10\sqrt{\frac{2}{4.9}} \text{ sec.}$$

2. Techniques of Integration

Technique 1: Algebraic simplification

Problem Find an anti-derivative

$$F(x) = \int (x^2+1)(x^4-2)\,dx .$$

Soln

$$(x^2+1)(x^4-2) = x^6 + x^4 - 2x^2 - 2$$

So

$$F(x) = \int x^6 + x^4 - 2x^2 - 2\,dx$$

$$= \frac{x^7}{7} + \frac{x^5}{5} - \frac{2x^3}{3} - 2x + C$$

↑ any constant

Problem Determine

$$F(x) = \int \frac{x^3 + x^2 - x - 1}{x - 1} \, dx .$$

Soln

$$x-1 \;\overline{)\; x^3 + x^2 - x - 1}$$

$$x^2 + 2x + 1$$

$$x^3 - x^2$$

$$2x^2 - x$$

$$2x^2 - 2x$$

$$x - 1$$

$$x - 1$$

So $\dfrac{x^3 + x^2 - x - 1}{x - 1} = x^2 + 2x + 1$

So

$$F(x) = \int x^2 + 2x + 1 \, dx$$

$$= \frac{x^3}{3} + x^2 + x + C .$$

Problem Find

$$F(x) = \int (x^2-1)^3 \, dx$$

Sol $(x^2-1)^3 =$

$(x^2-1)(x^2-1)(x^2-1)$

$= x^6 - 3x^4 + 3x^2 - 1$

So

$$F(x) = \int (x^2-1)^3 \, dx$$

$$= \int x^6 - 3x^4 + 3x^2 - 1 \, dx$$

$$= \frac{x^7}{7} - \frac{3x^5}{5} + x^3 - x + C .$$