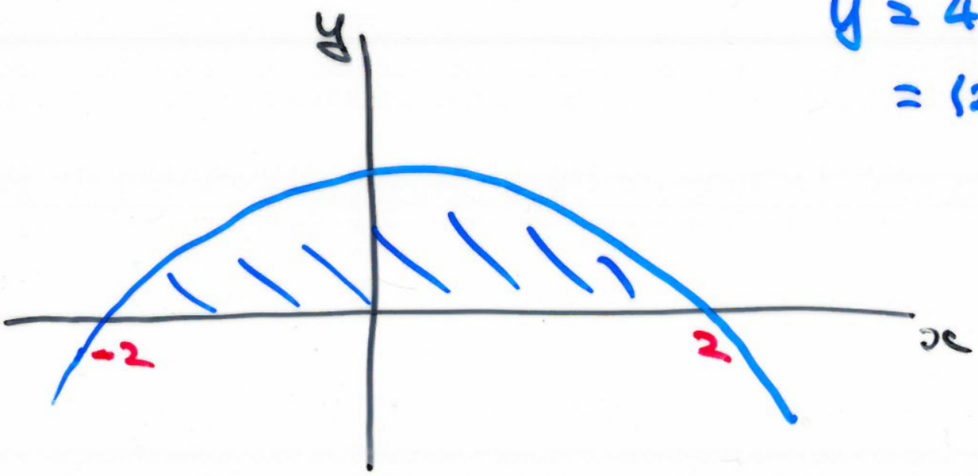
Problem Find the area of the region bounded by $y = 4 - x^2$ and the x-axis.

Sol^n

Step 1 Draw a picture of the region.

$y = 4 - x^2$
$= (2-x)(2+x)$



Since $y \geq 0$ for $x = -2$ and $x = +2$, the required area is just the integral

$$\text{area} = \int_{-2}^{2} 4 - x^2 \, dx$$

Step 2 we use

Fundamental Theorem of Calculus:

if $f(x) = \frac{d}{dx} F(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

In our problem, $f(x) = 4 - x^2$.

An anti-derivative of $f(x)$ is

$$F(x) = 4x - \frac{x^3}{3}$$

Since

$$\frac{d}{dx} F(x) = \frac{d}{dx}\left(4x - \frac{x^3}{3}\right) = 4 - x^2.$$

So, the area is

$$\text{area} = \int_{-2}^{2} 4 - x^2\,dx = F(2) - F(-2)$$

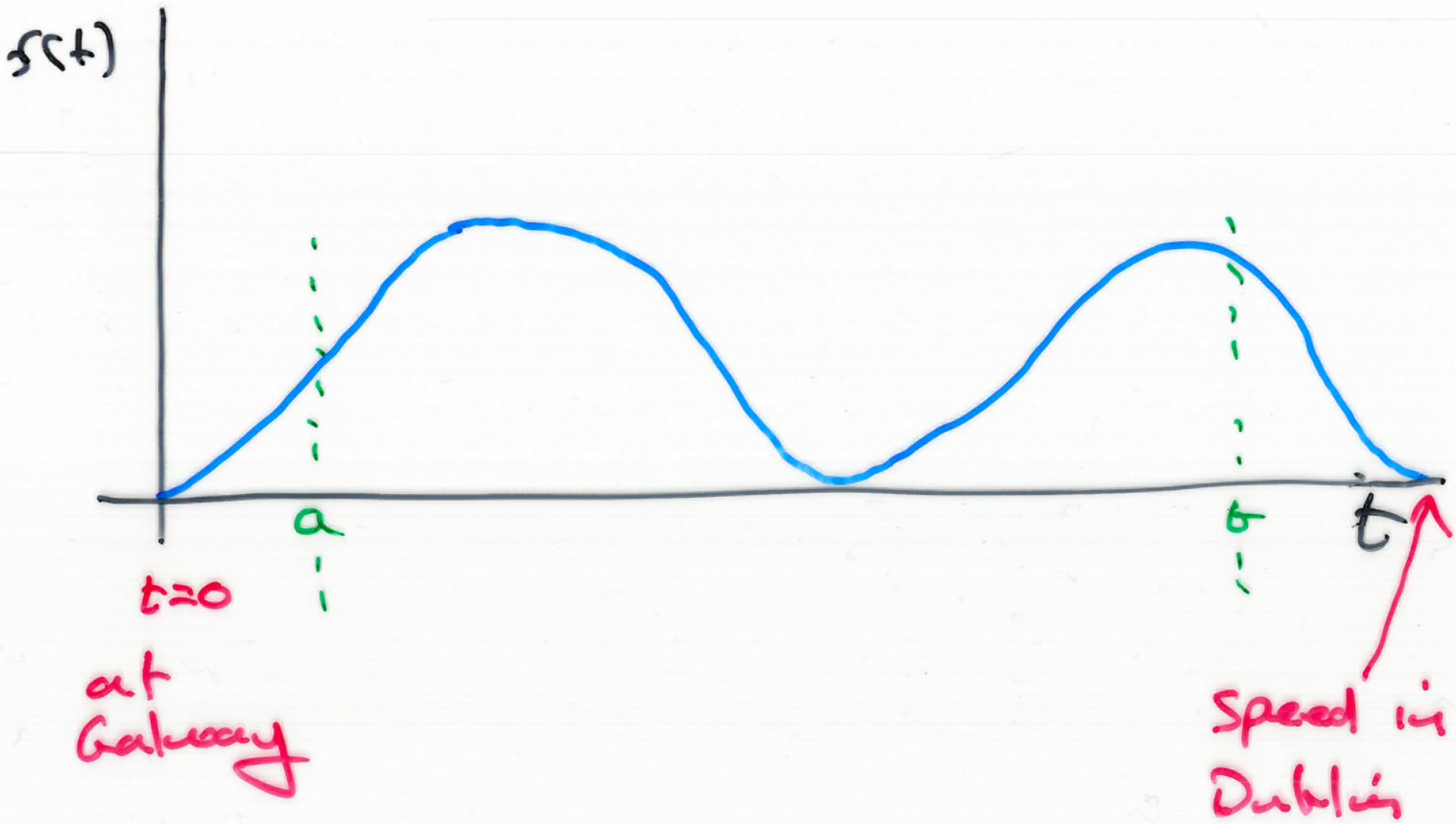
$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

$$= 16 - \frac{16}{3}.$$

Explanation of why the FTC holds

I'll use t in place of x, and $f(t)$, $F(t)$ in place of $f(x)$, $F(x)$.

Imagine that a train is travelling with speed $f(t)$ at time t. Maybe we have

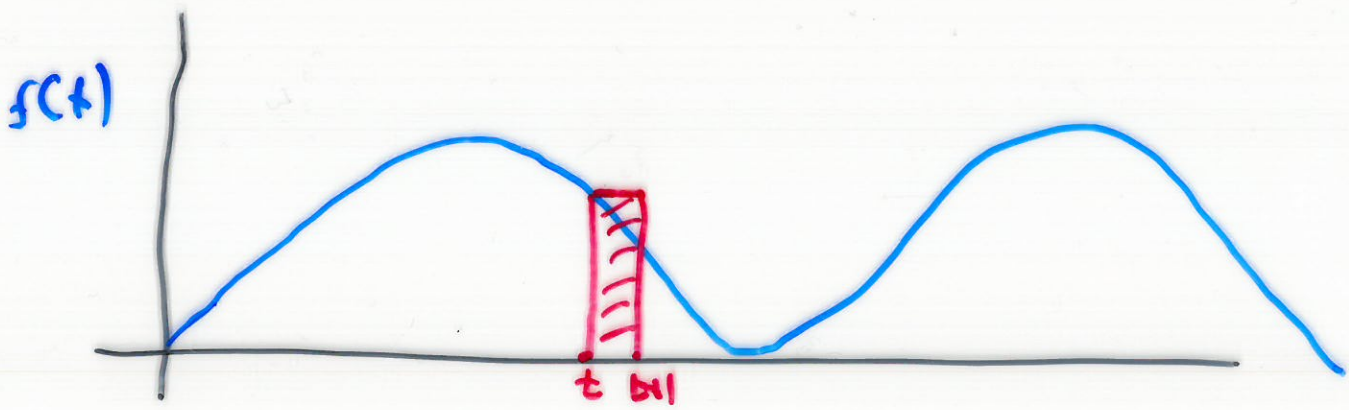


Imagine that the mileometer is broken on the train, but the speedometer is fine, and that the driver has an accurate watch.

The driver wants to approximate the distance travelled between time $t=a$ and time $t=b$.

<u>Question</u>: How can she do this?

<u>Answer</u>: Measure the speed every minute, and assume that the speed is roughly constant over each minute interval.



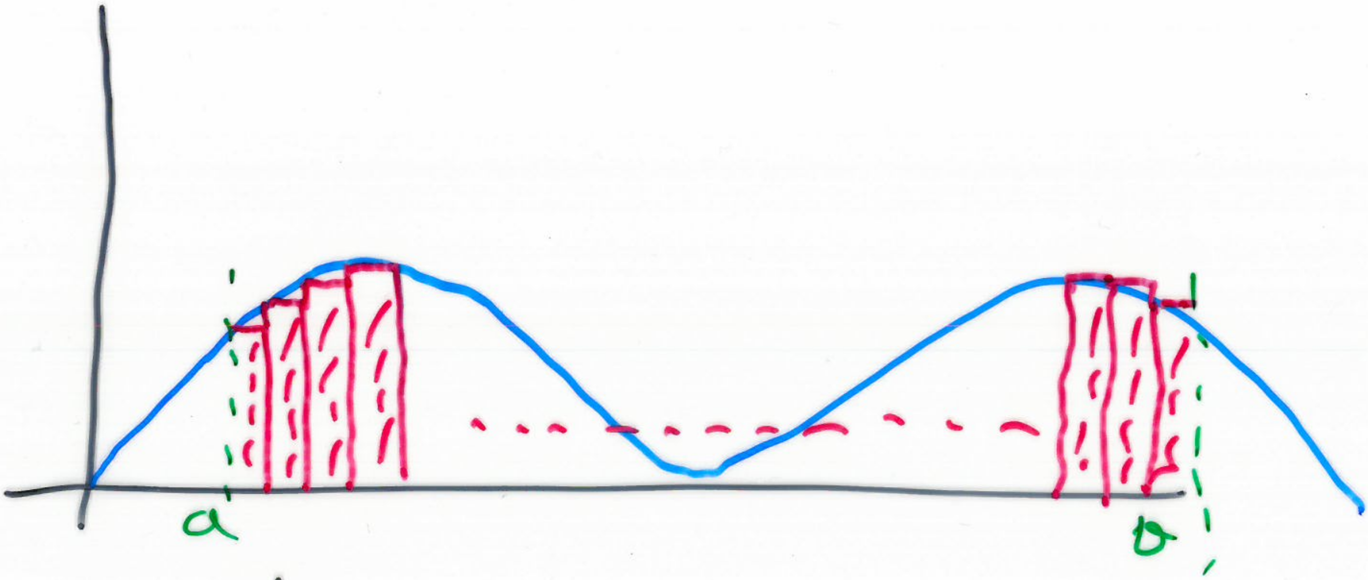
Distance travelled between t and $t+1$ minutes

$\simeq$

speed $\times$ time $= f(t) \times 1$

$=$ area of the box above.

So the total distance travelled from time $t=a$ to time $t=b$



is roughly:

Distance travelled $\simeq$ area of boxes $\simeq$ area between curve $y=f(t)$ and x-axis

$$= \int_a^b f(t)\,dt$$

1

if $f(t)$ = speed = rate of change of distance at time t.

What function $F(t)$ has

$$\frac{d}{dt}F(t) = f(t)\ ?$$

Answer: $F(t)$ = distance travelled by time t.

So distance travelled between $t=a$ and $t=b$ is equal to $F(b)-F(a)$. 2

From 1 & 2 we get

$$\int_a^b f(t)\,dt \approx F(b)-F(a).$$

The approximation becomes better if we measure speed every second.

Letting the time interval be ever smaller, in the limit we get

$$\int_a^b f(t)\,dt = F(b) - F(a).$$