1]

The floor function or

The greatest integer function

$y = \lfloor x \rfloor =$ the greatest integer $\leq x$

It means whatever number you put there it will round it down. observe that

$\lfloor 3/2 \rfloor = 1$, $\lfloor 3.142 \rfloor = 3$

$\lfloor -3.142 \rfloor = -4$, $\lfloor -3/2 \rfloor = -2$

ex. draw the graph of $y = \lfloor x \rfloor$ over

$-1 \leq x < 3$.

$-1 \leq x < 0 \Rightarrow y = \lfloor x \rfloor = -1$

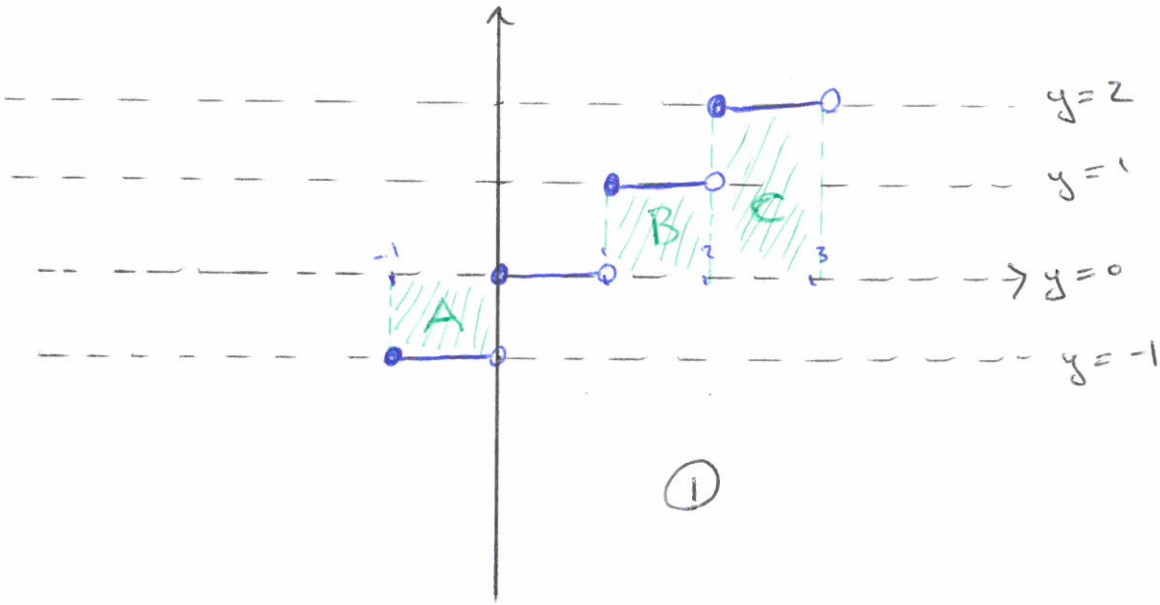
so over $-1 \leq x < 0$ the floor function equals $y = -1$ a line

$0 \leq x < 1 \Rightarrow y = \lfloor x \rfloor = 0$

$1 \leq x < 2 \Rightarrow y = \lfloor x \rfloor = 1$

$2 \leq x < 3 \Rightarrow y = \lfloor x \rfloor = 2$

2



①

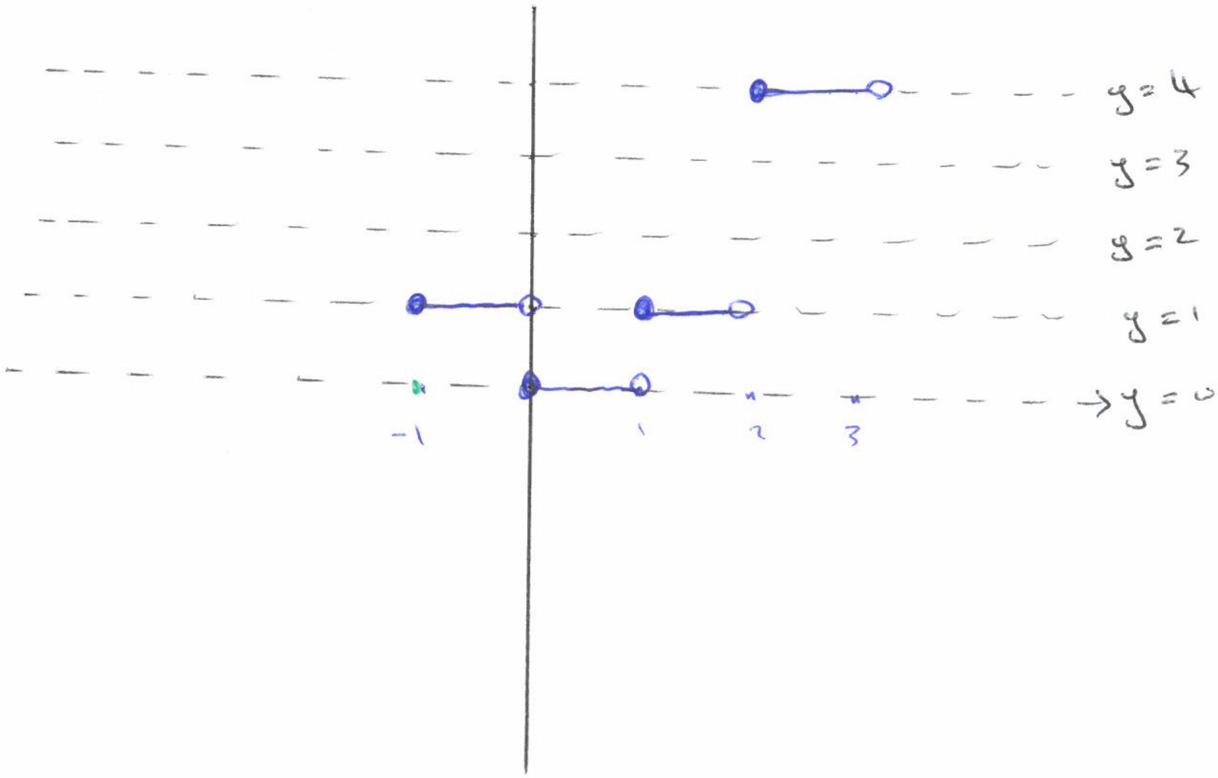
ex. The graph of $y = \lfloor x \rfloor^2$ over $-1 \leq x < 3$

$-1 \leq x < 0 \Rightarrow \lfloor x \rfloor = -1 \Rightarrow y = \lfloor x \rfloor^2 = 1$

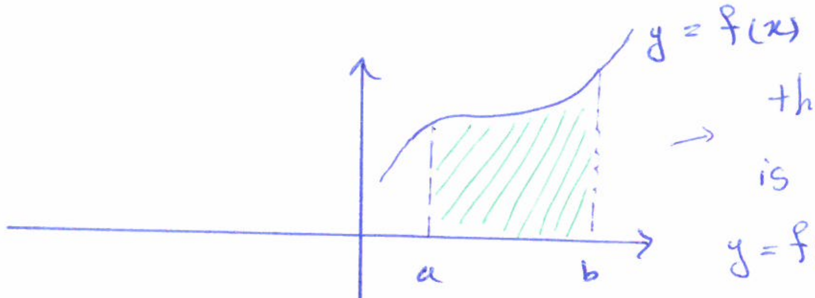
$0 \leq x < 1 \Rightarrow \lfloor x \rfloor = 0 \Rightarrow y = \lfloor x \rfloor^2 = 0$

$1 \leq x < 2 \Rightarrow \lfloor x \rfloor = 1 \Rightarrow y = \lfloor x \rfloor^2 = 1$

$2 \leq x < 3 \Rightarrow \lfloor x \rfloor = 2 \Rightarrow y = \lfloor x \rfloor^2 = 4$



3



→ the green region is bounded by $y = f(x)$, $y = 0$, $x = a$ and $x = b$.

$$\int_a^b f(x)\,dx = \text{The area of the green region}$$

ex. Evaluate $\displaystyle\int_{-1}^{3} \lfloor x \rfloor \, dx$

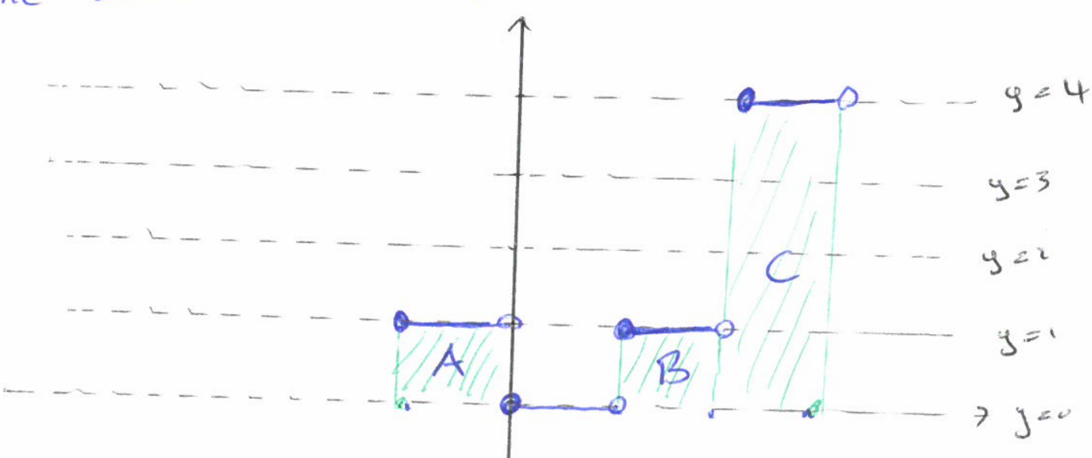
Look at ① in the second sheet:

area of the green region $= -A + B + C$

so $\displaystyle\int_{-1}^{3} \lfloor x \rfloor \, dx = -1 + 1 + 2 = 2$

ex. Evaluate $\displaystyle\int_{-1}^{3} \lfloor x \rfloor^2 \, dx$

we know the graph of $y = \lfloor x \rfloor^2$ (the second graph in the second sheet) so



$$\int_{-1}^{3} \lfloor x \rfloor^2 \, dx = A + B + C$$
$$= 1 + 1 + 4 = 6$$

4)

By proposition 1, we know that

$$\int_{-1}^{3} 2\lfloor x\rfloor^2 dx = 2\int_{-1}^{3}\lfloor x\rfloor^2 dx$$

so $$I = \underbrace{\int_{-1}^{3} 1\,dx}_{4} + \underbrace{\int_{-1}^{3} x\,dx}_{4} + 2\underbrace{\int_{-1}^{3}\lfloor x\rfloor^2 dx}_{6}$$

therefore $I = 4 + 4 + 2\times 6 = 4 + 4 + 12 = \underline{\underline{20}}$

Def: we say $F(x)$ is an anti-derivative of $f(x)$ if $\frac{d}{dx}F(x) = f(x)$.

ex. Find anti-derivative of the following:

1) $f(x) = x+1$ (Actually the quesion is: what it could be the derivative of $x+1$)

so $F(x) = \frac{1}{2}x^2 + x + C$

2) $f(x) = x^2 + 1 \Rightarrow F(x) = \frac{1}{3}x^3 + x + C$

the anti-derarative of $x^2 = \frac{1}{3}x^3$ → the anti-derivative of x^n is $\frac{1}{n+1}x^{n+1}$

" " " " $1 = x$

in general

so " " " " $x^2+1 = \frac{1}{3}x^3 + x + C$

So we write

$$\int x^2+1 \, dx = \frac{1}{3}x^3 + x + C$$

3) $f(x) = \sin x \implies F(x) = -\cos x + C$

so we write $\int \sin x \, dx = -\cos x + C$

4) $f(x) = \cos x \implies F(x) = \sin x + C$

so we write $\int \cos x \, dx = \sin x + C$

5) $f(x) = 2 + 3x^2 + 5\sin(x)$

$\Rightarrow F(x) = 2x + x^3 + (-5\cos(x))$

Fundamental Theorem of calculus:

Suppose that $F(x)$ is an anti-derivative of $f(x)$. Then

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

6) ex. Evaluate $I = \int_{-2}^{2} 1 + 2x^2 + x^3 \, dx$

$$I = \underbrace{\int_{-2}^{2} 1 \, dx}_{①} + 2 \underbrace{\int_{-2}^{2} x^2 \, dx}_{②} + \underbrace{\int_{-2}^{2} x^3 \, dx}_{③}$$

① the anti-derivative of $f(x) = 1$ is $F(x) = x$

so by the theorem: $\int_{-2}^{2} 1 \, dx = F(2) - F(-2)$

$$= 2 - (-2) = 4$$

② the anti-derivative of $f(x) = x^2$ is

$F(x) = \frac{1}{3}x^3$ so

$$\int_{-2}^{2} x^2 \, dx = F(2) - F(-2) = \frac{1}{3}(2)^3 - \frac{1}{3}(-2)^3$$

$$= \frac{8}{3} + \frac{8}{3} = \frac{16}{3}$$

③ the anti-derivative of $f(x) = x^3$ is

$F(x) = \frac{1}{4}x^4$ so

$$\int_{-2}^{2} x^3 \, dx = F(2) - F(-2) = \frac{1}{4}(2)^4 - \frac{1}{4}(-2)^4 = \underline{\underline{0}}$$

so $I = 4 + 2 \times \frac{16}{3} + 0 = 4 + \frac{32}{3} = \frac{12 + 32}{3} = \frac{44}{3}$