What will the world's population be in one hundred years time?

Last time: assumed the population $y(t)$ satisfies the <u>Malthusian Law</u>

$$\frac{dy}{dt} = k y \qquad (*)$$

Solutions to (*) have the form

$$y = A e^{kt}$$

form some constant A.

From real data we estimated the constants A, k. The resulting function y agrees well with data for the period 1700 – 1960.

But the function y is not so good for future/current periods.

The Malthusian Law ignores competition for resources.

The Logistic Model of population is

$$\frac{dy}{dt} = ky - ly^2 \qquad (**)$$

where k, l are constants, and l is very small compared to k. For small y the term ly^2 is insignificant and $(**)$ will be similar to $(*)$. But for larger y the term ly^2 will be important, and it will slow down the growth rate.

we need to:

1) find the general solutions to (**)

2) Then use real data to determine k, l.

But how do we solve (**)?

Separable Differential Equations

A differential equation is separable if it has the form

$$f(y) \frac{dy}{dt} = g(t)$$

for functions $f(y)$, $g(t)$.

Example

$$y^2 \frac{dy}{dt} = t^2$$

is a separable diff. equation.

Example

$$\frac{dy}{dt} = ky - \ell y^2$$

is a separable diff. equation since it can be rewritten

$$\frac{1}{ky - \ell y^2} \frac{dy}{dt} = 1$$

How do we solve a separable equation

$$f(y) \frac{dy}{dt} = g(t) \quad ?$$

Such an equation can be rewritten as

$$\frac{d}{dt} F(y) = g(t)$$

where $F(y)$ is any anti-derivative of $f(y)$

Consequenty

$$\boxed{F(y) = \int g(t)\,dt + c}$$

<u>Escample</u> Solve

$$y^2 \frac{dy}{dt} = t^2 .$$

$$\int y^2\,dy = \int t^2\,dt + c$$

$$\frac{1}{3} y^3 = \frac{1}{3} t^3 + c$$

$$y^3 = t^3 + c$$

$$y = (t^3 + c)^{\frac{1}{3}} .$$

Example Solve

$$e^y \frac{dy}{dt} - t - t^3 = 0.$$

Sol^n

$$e^y \frac{dy}{dt} = t + t^3$$

$$\int e^y \, dy = \int t + t^3 \, dt + C$$

$$e^y = \frac{t^2}{2} + \frac{t^4}{4} + C$$

$$\ln(e^y) = \ln\left(\frac{t^2}{2} + \frac{t^4}{4} + C\right)$$

$$y = \ln\left(\frac{t^2}{2} + \frac{t^4}{4} + C\right)$$

Example Solve

$$\frac{dy}{dt} = 1 + y^2 \quad , \quad y(0) = 0 .$$

Sol^n

$$\frac{1}{1+y^2}\frac{dy}{dt} = 1$$

$$\int \frac{1}{1+y^2}\, dy = \int 1 . dt + C$$

Let $y = \tan\theta$

$dy = \sec^2\theta \; d\theta$

$$\int \frac{1}{1+\tan^2\theta} \sec^2\theta \; d\theta = \int dt + C$$

$$\int d\theta = \int dt + C$$

$$\theta = t + C$$

Now $t=0, y=0 \Rightarrow 0 = \tan\theta \Rightarrow \theta = 0$

Then

$$0 = 0 + C.$$

So $C = 0$.

Therefore

$$\theta = t$$

and

$$\boxed{y = \tan(t)}$$

is our solution.