Differential Equations

An equation involving a derivative is a differential equation.

e.g.

$$\frac{dy}{dt} = k\,y \qquad (*)$$

where k is some constant and y some function of t.

1) Are there any solutions to $(*)$?

Consider for instance

$$y = e^{kt}.$$

Then $\frac{dy}{dt} = k e^{kt} = ky.$

So $y = e^{kt}$ is a solution of $(*)$.

Also,

$$y = 5e^{kt}$$

satisfies

$$\frac{dy}{dt} = k\,5e^{kt} = ky.$$

We see that

$$y = Ae^{kt}$$

is a solution to (*) for any constant A.

These are the only solutions to (*). To see this, suppose $y = y(t)$ and $z = z(t)$ both satisfy equation (*). Then

$$\frac{d}{dt}\left(\frac{y}{z}\right) = \frac{z'y - y'z}{z^2}$$

$$= \frac{kzy - kyz}{z^2}$$

$$= 0.$$

So $\frac{y}{z}$ is constant, say

$\frac{y}{z} = A$. So $y = Az$.

Problem A cup of coffee in a room at 20°C ~~cools~~ cools from 80°C to 50°C in five minutes. How long will it take to cool to 40°C?

Sol^n

Newton: A hot object cools at a rate proportional to the excess of its temperature above room temperature.

Let's introduce

$y(t)$ = temperature of coffee at time t.

$y(0) = 80$

$y(5) = 50$

Question is : for what t do we have $y(t) = 40$.

Newton?

$$\frac{dy}{dt} = k(y-20)$$

Consider $z = y - 20$

$z(0) = 60$

$z(5) = 30$

$$\frac{dz}{dt} = \frac{d}{dt}(y-20) = \frac{dy}{dt}$$

$$= k(y-20) = kz$$

Hence

$$\frac{dz}{dt} = kz$$

The solutions to this diff. equation are of the form

$$z = Ae^{kt}$$

Our question is: for what value of t do we have $z(t) = 40-20 = 20$.

We know

$$z = Ae^{kt}.$$

Continued tomorrow!