Recall We have

$$\frac{p(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} =$$

$$\frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \cdots + \frac{A_n}{x-a_n}$$

where $\deg(p(x)) < n$, and the $a_1, a_2, \ldots, a_n$ are distinct.

Comment 3

If the a_i are not all distinct, then the following method works.

Problem Find the indefinite integral

$$I = \int \frac{1}{x^3+x^2-x-1}\,dx$$

Solution

$$f(x) = x^3 + x^2 - x - 1$$

$f(1) = 1+1-1-1 = 0$, so $(x-1)$ divides $f(x)$

$f(-1) = -1+1+1-1 = 0$, so $(x+1)$ divides $f(x)$

$$x^3 + x^2 - x - 1 = (x-1)(x+1)^2$$
$$(x^2-1)(x+1)$$

$$\frac{1}{x^3+x^2-x-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Need to find A, B, C :

$$\frac{1}{x^3+x^2-x-1} = \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

$$1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$$

$x=1$: $\quad 1 = 4A \qquad A = \frac{1}{4}$

$x=-1$: $\quad 1 = -2C \qquad C = -\frac{1}{2}$

$x=0$: ~~1 = A~~

$1 = \frac{1}{4} - B + \frac{1}{2}$, $\quad B = -\frac{1}{4}$

so

$$I = \int \frac{1}{x^3+x^2-x-1}\,dx$$

$$= \frac{1}{4}\int \frac{1}{x-1}\,dx - \frac{1}{4}\int \frac{1}{x+1}\,dx - \frac{1}{2}\int \frac{1}{(x+1)^2}\,dx$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\int \frac{1}{u^2}\,du$$

where $u = x+1$

$dx = dx$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| + \frac{1}{2(x+1)} + C.$$

Comments 4 & 5

It might be the case that $\frac{p(x)}{q(x)}$ is such that $q(x)$ has quadratic factors of the form

$$ax^2 + bx + c.$$

In such case we would need to involve partial fractions of the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Random example

Find $I = \int \frac{1}{x^2+4}\,dx$.

Sol^n Let $x = 2\tan\theta$

$$dx = 2\sec^2\theta\,d\theta$$

$$I = \int \frac{1\,(2\sec^2\theta)\,d\theta}{4\tan^2\theta + 4}$$

$$I = 2\int \frac{\sec^2\theta \; d\theta}{4(\tan^2\theta + 1)}$$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{2}\int d\theta$$

$$= \frac{1}{2}\theta + C$$

where $\frac{x}{2} = \tan\theta$.