Technique 6 : partial fractions

$$\frac{2}{3} + \frac{4}{5} = \frac{2.5 + 4.3}{3.5} = \frac{22}{15}$$

$$\frac{2}{(x-1)} - \frac{1}{x+2} = \frac{2(x+2) - (x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}$$

$$\frac{2}{(x-1)} + \frac{x+3}{(x-1)^2} = \frac{2(x-1) + x+3}{(x-1)^2} = \frac{3x+1}{x^2-2x+1}$$

sum of partial fractions = fractions

<u>Problem</u> find $I = \int \frac{x+5}{x^2+x-2}\, dx$

<u>Sol^n</u>

$$I = \int \left(\frac{2}{x-1} - \frac{1}{x+2} \right) dx$$

$$= 2\int \frac{1}{x-1}\, dx - \int \frac{1}{x+2}\, dx$$

$$= 2 \ln|x-1| - \ln|x+2| + C.$$

But: given a fraction how do we express it as partial fractions?

Answer: involves five comments

Comment 1 using long division of polynomials if necessary, any fraction can be expressed in the form

$$p(x) + \frac{r(x)}{q(x)}$$

where $p(x)$, $q(x)$, $r(x)$ are polynomials and

$$\text{degree}(r(x)) < \text{degree}(q(x)).$$

Example

Find

$$I = \int \frac{x^3+x}{x-1}\,dx$$

Sol^n

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x-1 & x^3 + x \\
 & \underline{x^3 - x^2} \\
 & \quad x^2 + x \\
 & \quad \underline{x^2 - x} \\
 & \qquad 2x \\
 & \qquad \underline{2x - 2} \\
 & \qquad\qquad 2
\end{array}
$$

So $$\frac{x^3+x}{x-1} = x^2+x+2 + \frac{2}{x-1}$$

$$I = \int x^2+x+2+\frac{2}{x-1}\,dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C.$$

Comment 2 If a fraction is such that degree of top < degree of bottom, and if the bottom is a product of distinct linear factors then the following method will work.

Problem Find $I = \int \frac{8x+1}{2x^2-x-1} \, dx$

Soln

$$\frac{8x+1}{2x^2-x-1} = \boxed{\frac{8x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}}$$

for some constants A, B.

So

$$\frac{8x+1}{(2x+1)(x-1)} = \frac{A(x-1) + B(2x+1)}{(2x+1)(x-1)}$$

so

$$8x+1 = A(x-1) + B(2x+1)$$

$x=1: \quad 9 = 3B, \quad B = 3$

$x = -\frac{1}{2}: \quad -3 = -\frac{3}{2}A, \quad A = 2$

So $I = \int \frac{2}{2x+1}\,dx + 3\int \frac{1}{x-1}\,dx$

$= \ln|2x+1| + 3\ln|x-1| + C.$