Tutorials:

Wednesday 11 IT20? (usual place)

Wednesday 3 IT20? (")

Monday 3 SUMS

Technique 5: Trigonometric Substitutions

Problem Evaluate

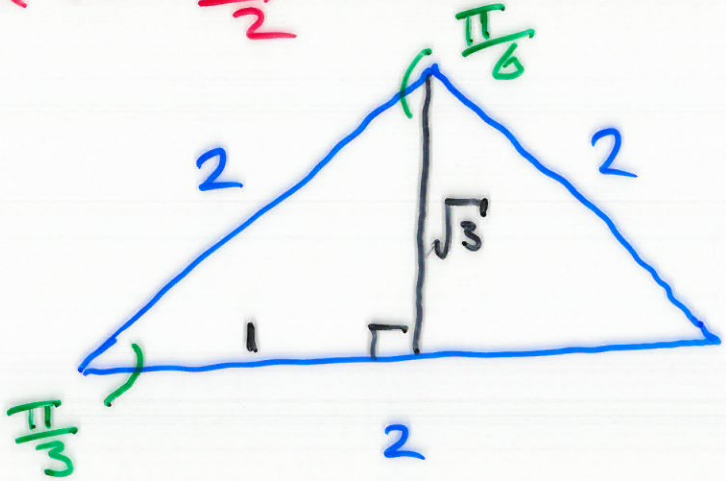
$$I = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\sqrt{1-x^2}}{x^2}\,dx$$

Soln Let $x = \sin\theta$

$dx = \cos\theta\, d\theta$

when $x = \frac{1}{2}$ we have $\theta = \frac{\pi}{6}$

when $x = \frac{\sqrt{3}}{2}$ we have $\theta = \frac{\pi}{3}$



So $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{1-\sin^2\theta}}{\sin^2\theta} \cos\theta \, d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2\theta}{\sin^2\theta} \, d\theta$

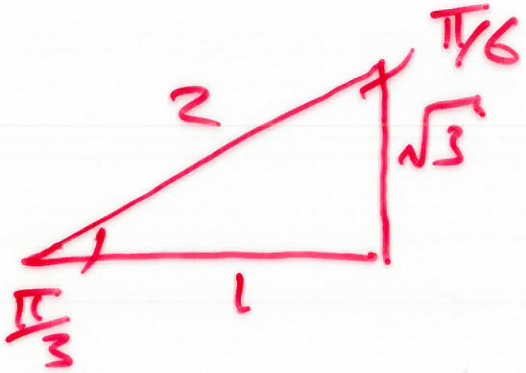
$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\sin^2\theta}{\sin^2\theta} \, d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin^2\theta} - 1 \, d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec}^2\theta - 1 \, d\theta$

$= -\cot\theta - \theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$$= -\cot\frac{\pi}{3} - \frac{\pi}{3} - \left(-\cot\frac{\pi}{6} - \frac{\pi}{6}\right)$$



$$= -\cot\frac{\pi}{3} - \frac{\pi}{6} + \cot\frac{\pi}{6}$$

$$= -\frac{1}{\sqrt{3}} - \frac{\pi}{6} + \sqrt{3} \quad .$$

Table of trig. substitutions

Expression	substitution	identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\cos^2\theta = \sin^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Problem Find

$$I = \int \frac{1}{x^2\sqrt{x^2+4}}\,dx$$

Sol^n Let $x = 2\tan\theta$

$$dx = 2\sec^2\theta\,d\theta$$

$$I = \int \frac{1 \cdot 2\sec^2\theta\,d\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}$$

$$= \frac{1}{4} \int \frac{\sec^2\theta \, d\theta}{\tan^2\theta \cdot 2 \cdot \sqrt{\tan^2\theta + 1}}$$

$$= \frac{1}{4} \int \frac{\sec^2\theta \, d\theta}{\tan^2\theta \cdot \sec\theta}$$

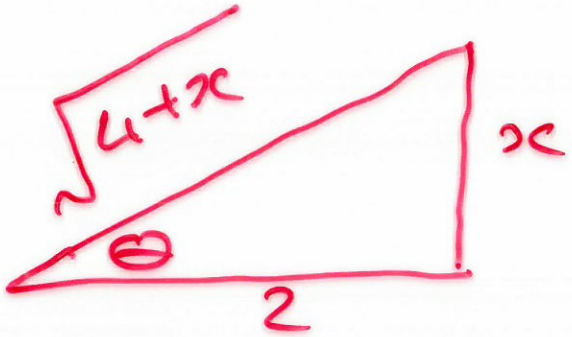
$$= \frac{1}{4} \int \frac{\sec\theta}{\tan^2\theta} \, d\theta$$

$$= \frac{1}{4} \int \frac{\cos^2\theta}{\cos\theta \cdot \sin^2\theta} \, d\theta$$

$$= \frac{1}{4} \int \frac{\cos\theta}{\sin^2\theta} \, d\theta$$

$$= \frac{1}{4} \int \cos\theta \cdot \sin^{-2}\theta \, d\theta$$

$$I = -\frac{1}{4} \frac{1}{\sin\theta} + C$$



$$I = 2 - \frac{1}{4} \frac{1}{\left(\frac{x}{\sqrt{4+x}}\right)} + c$$

$$= - \frac{\sqrt{4+x}}{4x} + c .$$