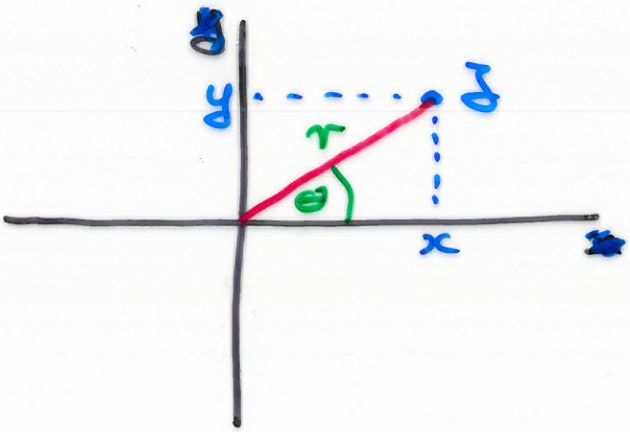
Recall

$z = x + iy$ $\qquad i^2 = -1$



$|z| = r = \sqrt{x^2+y^2}$

$\text{Arg}(z) = \theta$, $x = r\cos(\theta)$

$y = r\sin(\theta)$

So $z = r(\cos(\theta) + i\sin(\theta))$.

Problem Find $|z|$ and $\text{Arg}(z)$ for

$$z = \frac{3i^{30} - i^{19}}{2i - 1}$$

Soln Let's first express z in form

$z = x + iy$, $x, y \in \mathbb{R}$.

$$z = \frac{3(i^2)^{15} - i(i^2)^9}{2i - 1}$$

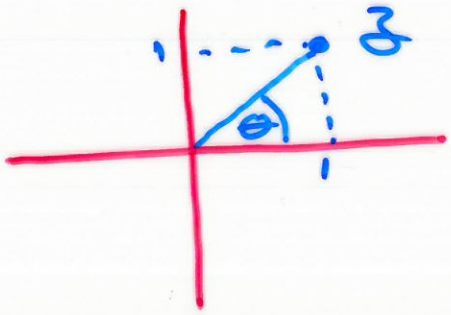
$$= \frac{(-3 + i)}{(-1 + 2i)} \cdot \frac{(-1 - 2i)}{(-1 - 2i)}$$

$$= \frac{3 + 2 - i + 6i}{1 + 4}$$

$$= \frac{5 + 5i}{5}$$

$$z = 1 + i$$

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$



$$\text{Arg}|z| = 45^\circ$$

$$= \frac{\pi}{4} \text{ rad}$$

Theorem For complex numbers w, z we have

$$|wz| = |w||z|$$

$$\text{Arg}(wz) = \text{Arg}(w) + \text{Arg}(z)$$

Proof Let

$$w = r(\cos\theta + i\sin\theta)$$

$$z = s(\cos\phi + i\sin\phi)$$

So $\text{Arg}(w) = \theta$, $|w| = r$

$\text{Arg}(z) = \phi$, $|z| = s$

now

$$wz = r(\cos\theta + i\sin\theta) \cdot s(\cos\phi + i\sin\phi)$$

$$= rs(\cos\theta\cos\phi - \sin\theta\sin\phi + i\{\sin\theta\cos\phi + \sin\phi\cos\theta\})$$

$$= rs(\cos(\theta+\phi) + i\sin(\theta+\phi)).$$

So

$$|wz| = rs = |w||z|$$

$\text{Arg}(wz) = \theta + \phi = \text{Arg}(w) + \text{Arg}(z)$.

III

Problem Evaluate

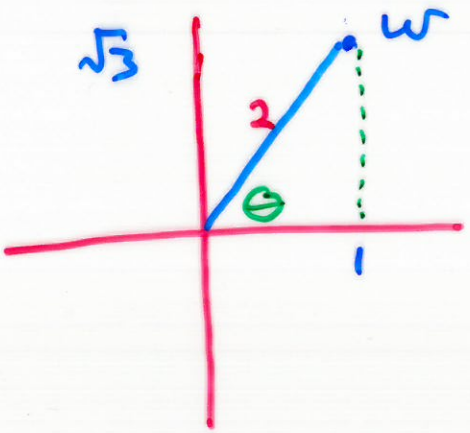
$$\left(\frac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i}\right)^{10}$$

Sol^n Let $w = 1+\sqrt{3}\,i$

$$z = 1-\sqrt{3}\,i$$

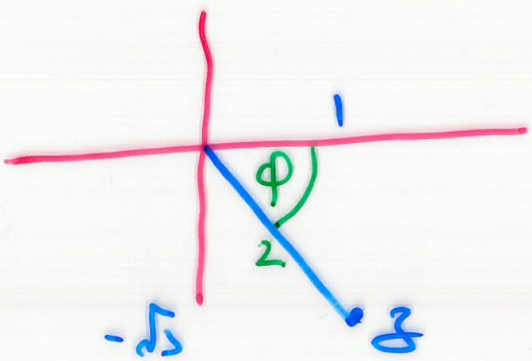
we need to calculate

$$\left(\frac{w}{z}\right)^{10} = \left(w\,z^{-1}\right)^{10} = w^{10}\,z^{-10}$$



$$|w| = \sqrt{1+3} = 2$$

$$\theta = 60^\circ$$



$$\phi = -60^\circ$$

$$\left(\frac{w}{z}\right)^{10} = w^{10} z^{-10}$$

$$= \left(2(\cos 60° + i \sin 60°)\right)^{10} \cdot \left(2(\cos -60° + i \sin -60°)\right)^{-10}$$

$$= 2^{10}(\cos 60° + i \sin 60°)\, 2^{-10} \cdot (\cos -60° + i \sin -60°)^{-10}$$

$$= (\cos 600 + i \sin 600) \cdot (\cos 600 + i \sin 600)$$

$$= (\cos 1200° + i \sin 1200°)$$

$$= \cos 120 + i \sin 120$$

$$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$