Algebra Topic 2 : Complex Numbers

$\mathbb{N} = \{1, 2, 3, 4, \cdots\}$ natural numbers

$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \cdots\}$ integers

An equation such as

$$2x - 3 = 0$$

has no integer solution. To allow for solutions we introduce

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$$

we add rational numbers

$$\frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq}$$

and multiply them

$$\frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}$$

The equation

$$2x - 3 = 0$$

has a solution $x = \frac{3}{2} \in \mathbb{Q}$.

The equation $x^2 - 2 = 0$ has no rational solution.

<u>Theorem</u> $\sqrt{2}$ is not in $\mathbb{Q}$.

<u>Proof</u>

Suppose $\sqrt{2} \in \mathbb{Q}$. Then

$$\sqrt{2} = \frac{m}{n}$$ with $m, n \in \mathbb{Z}$ and $\mathrm{hcf}(m,n) = 1$.

So

$$2 = \frac{m^2}{n^2}$$

and

$$2n^2 = m^2 .$$

So m^2 is even.

Since the square of any odd number

is odd, we conclude that m must be even.

So

$$m = 2M \qquad M \in \mathbb{Z}.$$

We can thus rewrite

$$2n^2 = m^2$$

as

$$2n^2 = (2M)^2$$

or

$$2n^2 = 4M^2$$

or

$$n^2 = 2M^2.$$

So n^2 is even, and thus so too is n.

But this contradicts $\mathrm{hcf}(m,n) = 1$.

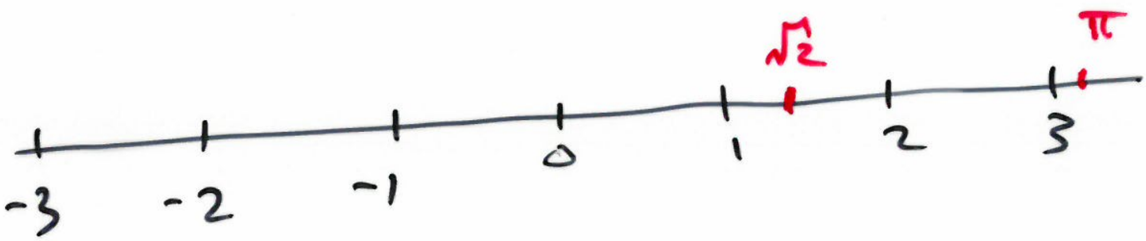
Hence, $\sqrt{2}$ is <u>not</u> rational.

QED.

To allow for solutions to $x^2 - 2 = 0$ we introduce:

$\mathbb{R}$ = real numbers.

we picture $\mathbb{R}$ as an infinite line



The equation

$$x^2 + 1 = 0$$

has no real solution!

We introduce the symbol

$$i$$

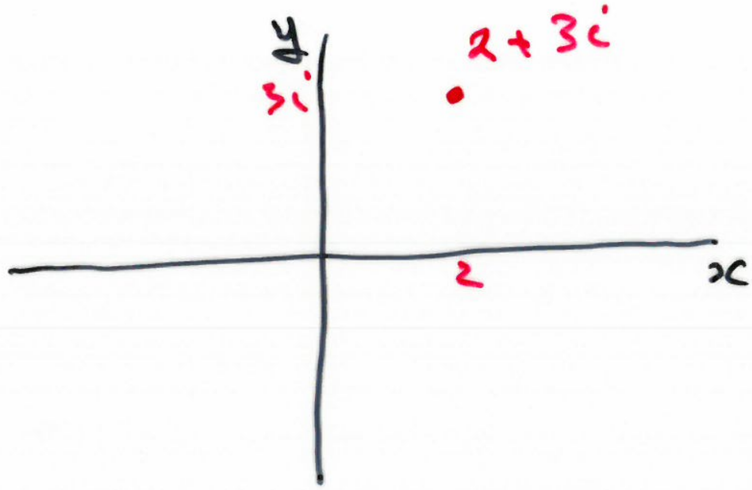
and assume

$$i^2 = -1 .$$

We then let

$$\mathbb{C} = \{ x + iy \ : \ x, y \in \mathbb{R} \} .$$

We picture $\mathbb{C}$ as a plane



<u>Addition of complex numbers</u>

$$(2+3i) + (-1+4i) = (1+7i)$$

<u>Subtraction</u>

$$(2+3i) - (-1+4i) = (3-i)$$

Multiplication

$$(2+3i)\cdot(-1+4i) =$$

$$2(-1) + 2(4i) + (3i)(-1) + (3i)(4i)$$

$$= -2 + 8i - 3i + 12i^2$$

$$= -2 + 8i - 3i - 12$$

$$= -14 - 5i$$

Tomorrow : division !