Problem Solve

$$w - 2x + y - z = 4$$
$$2w - 3x + 2y - 3z = -1$$
$$3w - 5x + 3y - 4z = 3$$
$$-w + x - y + 2z = 5$$

Soln

$$w - 2x + y - z = 4$$
$$x - z = -9 \qquad R_2 \to R_2 - 2R_1$$
$$x - z = -9 \qquad R_3 \to R_3 - 3R_1$$
$$-x + z = 9 \qquad R_4 \to R_4 + R_1$$

$$\underline{\underline{w}} - 2x + y - z = 4$$
$$\underline{\underline{x}} - z = -9$$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 + R_3$

Terminology: The first non-zero entries in the rows of the simplified system are called dependent variables

so w and x are dependent variables in this example.

So the solutions to our system are of the form

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -14+3\lambda-\mu \\ \lambda-9 \\ \mu \\ \lambda \end{pmatrix}$$

or

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -14 \\ -9 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

So our solutions form a "plane". More precisely, we have a <u>2-dimensional</u> <u>solution</u> <u>space</u>.

Comment: Different row operations could result in different independent variables.

<u>Theorem</u>: The <u>number</u> of independent variables does not depend on the choice of row operations.

The remaining variables are called independent variables

In our example y and z are independent.

Fact: we can assign any values to the independent variables.

Let's assign

$z = \lambda$

$y = \mu$

from R_2: $x - \lambda = -9$

$x = \lambda - 9$

from R_1:

$w - 2(\lambda - 9) + \mu - \lambda = 4$

$w = -14 + 3\lambda - \mu$.

Problem Find all values of k for which the following system has no solutions.

$$x + ky = 0$$
$$kx + 9y = 1.$$

Sol^n

$$x + ky = 0$$
$$(9-k^2)y = 1 \qquad R_2 \to R_2 - kR_1$$

There is no solution for $k = 3$ or $k = -3$ since we can't have $0y = 1$. There is a unique solution for any value of $k \neq 3, -3$.

MATRIX NOTATION

The linear system

$$
\begin{aligned}
x + 2y + 3z &= 10 \\
2x + 5y + 5z &= 21 \qquad (*) \\
3x + 8y + 6z &= 31
\end{aligned}
$$

Can be more succinctly expressed as follows:

$$
\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}
\begin{pmatrix} x \\ y \\ z \end{pmatrix}
=
\begin{pmatrix} 10 \\ 21 \\ 31 \end{pmatrix}
$$