Conclusion from last week:

A linear equation

$$ax + by + cz = d$$

determines a plane in $\mathbb{R}^3$.

Observation: A system of two linear equations

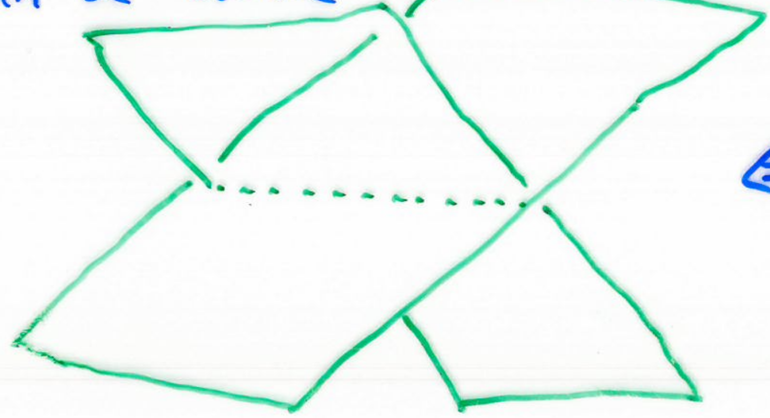
$$(*) \begin{cases} ax + by + cz = d \\ a'x + b'y + c'z = d' \end{cases}$$

can be regarded as two planes.

For such a system

<u>EITHER</u>

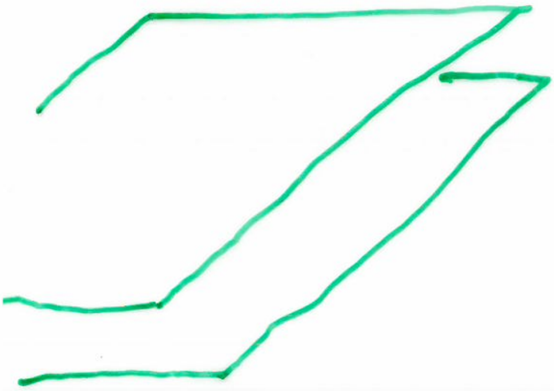
Two planes intersect in a line



$\Longleftrightarrow$ The system $(*)$ has infinitely many solutions (determined by one parameter)

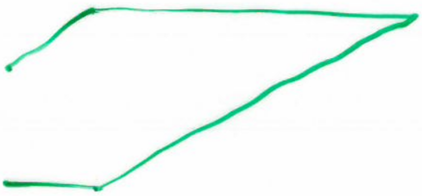
OR

The planes are parallel $\Longleftrightarrow$ The system (*) has no solution



OR

The planes are identical $\Longleftrightarrow$ The system (*) has infinitely many solutions (determined by two parameters)



Example

Solve the system of equations

$$3x + 5y + 7z = 15$$
$$x + y + z = 1$$

Sol 4

$$x + y + z = 1$$
$$3x + 5y + 7z = 15$$

Equivalent system $(R_2 \to R_2 - 3R_1)$

$$x + y + z = 1$$
$$2y + 4z = 12$$

Any solution

$$Q = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

to the system has the form

$$Q = \begin{pmatrix} -5+\lambda \\ 6-2\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Thus the system has as solutions all points on the line determined by

$P = \begin{pmatrix} -5 \\ 6 \\ 0 \end{pmatrix}$ and direction vector $v = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Higher dimensional equations

Example Solve the system

$$\left.\begin{aligned} w + 3x + 3y + 2z &= 1 \\ 2w + 6x + 9y + 5z &= 5 \\ -w - 3x + 3y &= 5 \end{aligned}\right\}$$

Sol^n The following system is equivalent. $\left(\begin{aligned} R_2 &\to R_2 - 2R_1 \\ R_3 &\to R_3 + R_1 \end{aligned} \right)$

$$\begin{aligned} w + 3x + 3y + 2z &= 1 \\ 3y + z &= 3 \\ 6y + 2z &= 6 \end{aligned}$$

now apply $R_3 \to R_3 - 2R_3$

$$w + 3x + 3y + 2z = 1$$

$$3y + z = 3$$

This system has the same ~~[illegible]~~ solutions as our original system.

Let's take

$$\boxed{z = \lambda}$$

$$3y + \lambda = 3$$

$$3y = 3 - \lambda \qquad \boxed{y = 1 - \tfrac{1}{3}\lambda}$$

Let's take

$$\boxed{x = \mu}$$

$$w + 3\mu + 3\left(1 - \tfrac{1}{3}\lambda\right) + 2\lambda = 1$$

$$w + 3\mu + 3 + \lambda = 1$$

$$\boxed{w = -2 - \lambda - 3\mu}$$

The solutions to our original system of equations have the form

$$Q = \begin{pmatrix} -2-\lambda-3\mu \\ \mu \\ 1-\frac{1}{3}\lambda \\ \lambda \end{pmatrix}$$

i.e.

$$Q = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

These solutions form a "plane in $\mathbb{R}^4$".