Past Exam Questions: MA121, MA123

2006-07, Question 7

a) Find the modulus and argument of $\omega = \dfrac{\sqrt{3}+i}{\sqrt{3}-i}$ and express ω^8 in the form $x + iy$ (x, y real numbers).

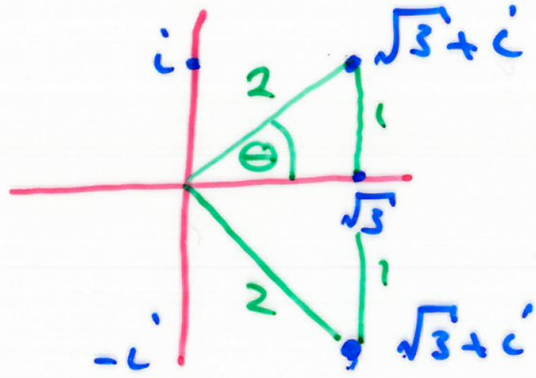
Sol^n

$$|\sqrt{3}+i| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$|\sqrt{3}-i| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$\text{Arg}(\sqrt{3}+i) = \frac{\pi}{6}$$

$$\text{Arg}(\sqrt{3}-i) = -\frac{\pi}{6}$$



$$|\omega| = \frac{|\sqrt{3}+i|}{|\sqrt{3}-i|} = \frac{2}{2} = 1$$

$$\text{Arg}(\omega) = \text{Arg}(\sqrt{3}+i) - \text{Arg}(\sqrt{3}-i)$$

$$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$$

$|w^8| = |w|^8 = 1^8 = 1$

$\text{Arg}(w^8) = 8\,\text{Arg}(w) = \frac{8\pi}{3} = \frac{2}{3}\pi$

$w^8 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

b) Show that $a = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ satisfies $a^6 = 1$ and write down all solutions of $z^6 = 1$ in terms of a. Hence or otherwise express $x^3 - 1$ as a product of (complex) linear factors.

$$a^6 = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^6 \underset{\text{De Moivre's Theorem}}{=} \left(\cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3}\right) = 1$$

Solutions of $z^6 = 1$ are

$a^0 = 1$

a

a^2

a^3

a^4

a^5

$x^3 - 1 = (x - a^0)(x - a^2)(x - a^4)$

c) Expand $(\cos\theta + i\sin\theta)^3$ and then show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

$$= (\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta)(\cos\theta + i\sin\theta)$$

$$= \cos^3\theta + 2i\sin\theta\cos^2\theta - \sin^2\theta\cos\theta$$
$$+ i\sin\theta\cos^2\theta - 2\sin^2\theta\cos\theta - i\sin^2\theta$$

$$= \cos^3\theta - \sin^2\cos\theta - 2\sin^2\theta\cos\theta$$
$$+ i(2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^2\theta).$$

Equating real parts:

$$\cos 3\theta = \cos^3\theta - 3\sin^2\cos\theta$$
$$= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$$
$$= 4\cos^3\theta - 3\cos\theta$$