Recall

$$z = x + iy$$

can be written in the form

$$r(\cos\theta + i\sin\theta)$$

where $r = |z|$

$\theta = \mathrm{Arg}(z)$.

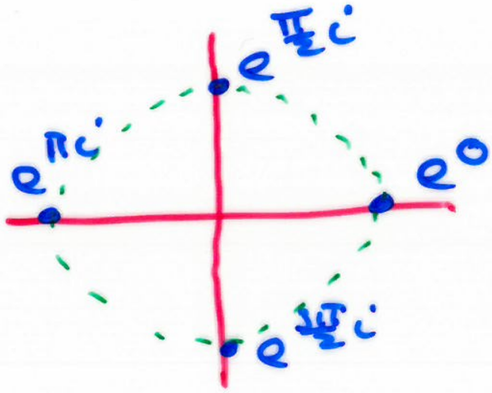
Notation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

so

$e^{0} = 1$ $\quad e^{\pi i} = -1$

$e^{\frac{\pi}{2}i} = i$ $\quad e^{\frac{3\pi}{2}i} = -i$



De Moivre's Theorem:

$$e^{i\theta} e^{i\phi} = e^{i\theta + i\phi} = e^{i(\theta+\phi)}$$

e.g. $e^{i\frac{\pi}{2}} \cdot e^{i\pi} = e^{i\frac{3\pi}{2}}$

$i \cdot (-1) = -i$

Let's list the 5th roots of unity, i.e. those complex numbers z satisfying $z^5 = 1$.

we have

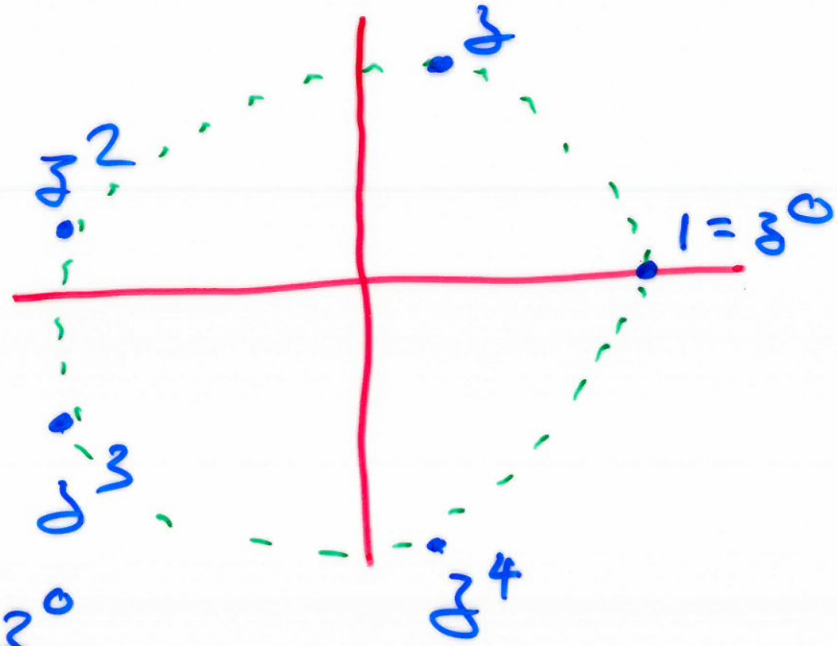
$$z = e^{\frac{2\pi}{5} i}$$

$$z^2 = e^{\frac{4\pi}{5} i}$$

$$z^3 = e^{\frac{6\pi}{5} i}$$

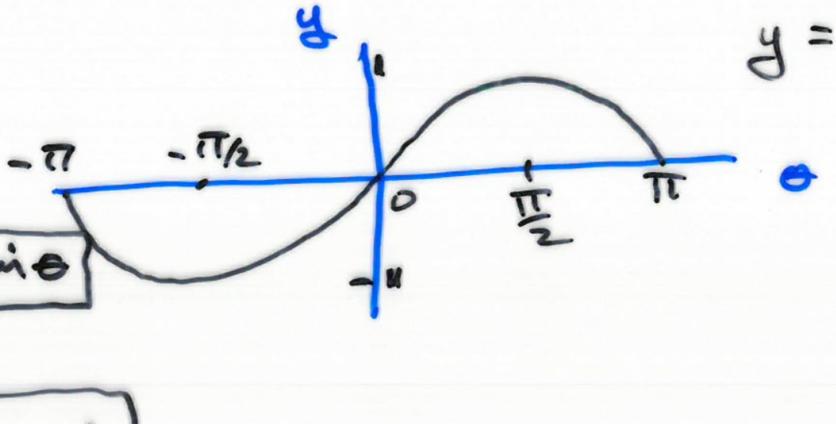
$$z^4 = e^{\frac{8\pi}{5} i}$$

$$z^5 = e^{0i} = 1 = z^0$$



Recall

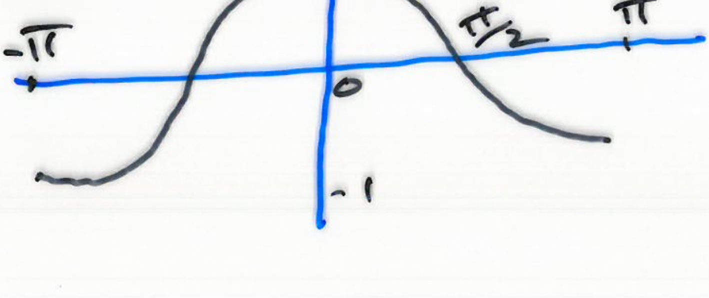
$y = \sin\theta$



$\boxed{\sin(-\theta) = -\sin\theta}$

$\boxed{\cos\theta = \cos(-\theta)}$

$y = \cos\theta$



Notation The complex conjugate

of

$$z = a + ib$$

is

$$\bar{z} = a - ib .$$

for $\boxed{z = e^{\frac{2\pi}{5}i} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}}$ — (1)

$$z^4 = e^{\frac{8\pi}{5}i} = e^{-\frac{2\pi}{5}i} = \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)$$

$\boxed{z^4 = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}}$ — (2)

So (1) & (2) say $\boxed{z^4 = \bar{z}}$

Exercise:

$$\boxed{z^2 = \bar{z}^3}$$

General fact: n^{th} roots of unity are either real $(1, -1)$ or come in complex conjugate pairs.

<u>useful algebra</u>

Let $z = a + ib$. Then

$$\boxed{z + \bar{z} = 2a}$$

$$\boxed{z\bar{z} = (a+ib)(a-ib) = a^2 + b^2}$$

Let's factorise $x^5 - 1$.

$$x^5 - 1 = (x - z^0)(x - z)(x - z^2)(x - z^3)(x - z^4)$$

$$= (x - 1)(x - z)(x - \bar{z})(x - z^2)(x - \bar{z^2})$$

$$= (x - 1)(x^2 - (z + \bar{z})x + 1)(x^2 - (z^2 + \bar{z^2})x + 1)$$

$$= (x - 1)(x^2 - 2\cos\tfrac{2\pi}{5}x + 1)(x^2 - 2\cos\tfrac{4\pi}{5}x + 1)$$