MA287: Tutorial Problems 2012-13

Tutorials: Tuesday, 6-7pm, Dillon
Thursday, 2-3pm, Tyndall

Tutor: Le Van Luyen

First Class Test: Monday 4th February (based on Problems I & II)

Second Class Test: Monday 4th March (based on Problems III & IV)

Third & Fourth Class Tests: Monday 25th March (based on Problems V, VI, VII)

Problems I

1.53 Express

$$\frac{2-3\mathbf{i}}{4-\mathbf{i}} \text{ and } (4+\mathbf{i})(3+2\mathbf{i})(1-\mathbf{i})$$

in the form $x+y\mathbf{i}$.

1.4 Prove that $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$ and $|z_1 z_2| = |z_1||z_2|$.

1.7 Prove that $|z_1+z_2| \le |z_1| + |z_2|$.

1.16 Express $2+2\sqrt{3}\mathbf{i}$ and $-\sqrt{6}-\sqrt{2}\mathbf{i}$ in polar form.

1.26 Evaluate

$$\left(\frac{1+\sqrt{3}\mathbf{i}}{1-\sqrt{3}\mathbf{i}}\right)^{10}.$$

1.30 Find the square roots of $-15-8\mathbf{i}$.

1.47 A number is called an *algebraic number* if it is a solution of a polynomial equation $a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$ where $a_0, a_1, \ldots, a_n$ are all integers. Prove that $4^{1/3} - 2\mathbf{i}$ is an algebraic number.

1.71 Describe and graph the locus represented by each of the following:

(a) $|z+2\mathbf{i}| + |z-2\mathbf{i}| = 6$,

(b) $\mathrm{Im}(z^2) = 4$.

1.128 Explain the fallacy: $-1 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$. Hence $1 = -1$.

1.48 Represent graphically the set of values of z for which

$$\left|\frac{z-3}{z+3}\right| = 2\ .$$

Problems II

2.9 Prove that $\sin^2 z + \cos^2 z = 1$.

2.12 Prove that $1 - \tanh^2 z = \text{sech}^2 z$.

2.3 A point P moves in a counterclockwise direction around a circle in the z plane having centre at the origin and radius 1. If the mapping function is $w = z^3$, show that when P makes one complete revolution, the image P' of P in the w plane makes three complete revolutions in a counterclockwise direction on a circle having center at the origin and radius 1.

2.86 Describe a Riemann surface S for the function $f(z) = z^3$ by determining: (i) a branch point, (ii) a branch line, (iii) the number of copies of $\mathbb{C}$ that are glued together, and the fashion in which they are glued.

2.85 Construct a Riemann surface for $f(z) = (z^2 + 1)^{\frac{1}{3}}$.

2.67 Show that $\overline{\sin z} = \sin \overline{z}$ and $\overline{\cos z} = \cos \overline{z}$.

2.91 Guess at a possible value for

$$\lim_{z \to 2+i} \frac{1-z}{1+z}.$$

2.102 Let $f(z) = (z^2 + 4)/(z - 2\mathbf{i})$ for $z \neq 2\mathbf{i}$ and $f(2\mathbf{i}) = 3 + 4\mathbf{i}$. Is $f(z)$ continuoius at $z = 2\mathbf{i}$?

2.44 Find a non-constant mapping function which maps the points $z = 0, \pm\mathbf{i}, \pm 2\mathbf{i}, \pm 3\mathbf{i}, ...$ of the z plane into the point $w = 1$ of the w plane.

Problems III

3.3 For $w = f(z) = (1+z)/(1-z)$ find dw/dz and determine where $f(z)$ is non-analytic.

3.7 Prove that $u = e^{-x}(x \sin y - y \cos x)$ is harmonic. Then find v such that $f(z) = u + \mathbf{i}v$ is analytic.

3.13 Prove that $d/dz(z^{1/2}) = 1/(2z^{1/2})$ bearing in mind that $z^{1/2}$ is a multivalued function.

3.24 Evaluate $\lim_{z \to 0}(\cos z)^{1/z^2}$.

3.28 Find the orthogonal trajectories of the family of curves in the xy plane which are defined by $e^{-x}(x \sin y - y \cos y) = \alpha$, α a real constant.

3.36 Suppose $A(x, y) = 2xy - \mathbf{i}x^2y^3$. Find $\text{grad}A$ and $\text{div}A$.

3.46 Determine the singular points of $f(z) = z/(z + \mathbf{i})$ and determine the derivative at all other points.

3.47 Verify that the real and imaginary parts of $f(z) = z^2 + 5\mathbf{i}z + 3 - \mathbf{i}$ satisfy the Cauchy-Riemmann equations.

3.53 Determine if $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. If it is, determine the conjugate harmonic v and express $u + \mathbf{i}v$ as an analytic function of z.

3.82 Locate and name the finite singularities of $f(z) = (z^2 - 3z)/(z^2 + 2z + 2)$.

3.86 Find the orthogonal trajectories of the family of curves $x^3y - xy^3 = \alpha$.

3.97 Find an equation for the line normal to the curve $x^2y = 2xy + 6$ at the point $(3, 2)$.

Problems IV

4.1 Evaluate

$$\int_{(0,3)}^{(2,4)} (2y + x^2)dx + (3x - y)dy$$

along the parabola $x = 2t$, $y = t^2 + 3$.

4.32 Evaluate

$$\int_{(0,1)}^{(2,5)} (3x + y)dx + (2y - x)dy$$

along the straight line joining $(0, 1)$ and $(2, 5)$.

4.36 Evaluate

$$\int_C (z^2 + 3z)dz$$

along the circle $|z| = 2$ from $(2, 0)$ to $(0, 2)$ in a counterclockwise direction.

4.22 Evaluate

$$\int_C \frac{dz}{(z - a)^n}$$

for $n = 2, 3, 4, \ldots$ where $z = a$ lies inside the simple closed curve C.

4.39 Evaluate

$$\int_C \overline{z}^2 dz$$

around the circle $|z| = 1$.

4.30 Give an example of a continuous, closed, non-intersecting curve that lies in a bounded region R but which has infinite length.

Problems V

5.1 Let $f(z)$ be analytic inside and on the boundary C of a simply connected region R. Let a be in the interior of R. Prove Cauchy's integral formula

$$f(a) = \frac{1}{2\pi \mathbf{i}} \int_C \frac{f(z)}{z-a}\, dz$$

in which C is assumed to have the positive direction.

5.30 Evaluate

$$\int_C \frac{e^z}{z-2}\, dz$$

where C is the circle $|z| = 3$ with positive direction.

5.32 Evaluate

$$\int_C \frac{e^{3z}}{z-\pi\mathbf{i}}\, dz$$

where C is the ellipse $|z-2| + |z+2| = 6$ with positive direction.

5.2 Evaluate

$$\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)}\, dz$$

where C is the circle $|z| = 3$ with the positive direction.

5.28 Let $F(z)$ be analytic inside and on a simple closed curve C with positive direction, except for a pole of order m at $z = a$ inside C. Prove that

$$\frac{1}{2\pi\mathbf{i}} \int_C F(z)\, dz = \lim_{a \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m F(z)\}.$$

5.29 Evaluate

$$\int_C \frac{e^z}{(z^2+\pi^2)^2}\, dz$$

where C is the circle $|z| = 4$ with positive direction.

5.49 A non-constant function $F(z)$ is such that $F(z+a) = F(z)$ and $F(z+bi) = F(z)$ where $a > 0, b > 0$ are given real constants. Prove that $F(z)$ cannot be analytic in the rectangle $0 \le x \le a$, $0 \le y \le b$.

Problems VI

6.1 Using the definition of a limit, prove that

$$\lim_{n\to\infty} (1 + \frac{z}{n}) = 1$$

for all $z \in \mathbb{C}$.

6.35 Using the definition of a limit, prove that

$$\lim_{n\to\infty} \frac{3n-2z}{n+z} = 3$$

for all $z \in \mathbb{C}$.

6.2 Prove that the series

$$z(1-z) + z^2(1-z) + z^3(1-z) + \cdots$$

converges for $|z| < 1$ and find its sum.

6.23 Expand

$$\log(\frac{1+z}{1-z})$$

in a Taylor series about $z = 0$.

6.26(a) Find the Laurent series of

$$\frac{e^{2z}}{(z-1)^3}$$

about the singularity $z = 1$. What type of singularity is it? What is the region of convergence?

6.26(a) Find the Laurent series of

$$\frac{z}{(z+1)(z+2)}$$

about the singularity $z = -2$.What type of singularity is it? What is the region of converge nce?

6.27 Expand

$$f(z) = \frac{1}{(z+1)(z+3)}$$

in a Laurent series valid for: (a) $1 < |z| < 3$, (b) $|z| > 3$, (c) $0 < |z+1| < 2$, (d) $|z| < 1$.

6.78 Expand

$$\frac{1}{1+z}$$

in a Taylor series about $z = 1$.

6.91 Expand

$$\frac{1}{z-3}$$

in a Laurent series valid for $|z| < 3$.

Problems VII

7.4 Find the residues of

$$f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)}$$

at all poles in th finite plane. Evaluate

$$\int_\Gamma \frac{z^2 - 2z}{(z+1)^2(z^2+4)}\, dz$$

where Γ is the circle $|z - 3\mathbf{i}| = 3/2$ with positive orientation.

7.4 Find the residues of

$$f(z) = \frac{e^z}{\sin^2 z}$$

at all poles in the finite plane.

7.6 Evaluate

$$\frac{1}{2\pi\mathbf{i}} \int_\Gamma \frac{e^{zt}}{z^2(z^2+2z+2)}\, dz$$

where Γ is the circle $|z| = 3$ with positive orientation.

6.9 Evaluate

$$\int_0^\infty \frac{1}{(x^6+1)}\, dx.$$

6.10 Evaluate

$$\int_{-\infty}^\infty \frac{x^2}{(x^2+1)^2(x^2+2x+2)}\, dx.$$