Gradient

Defn We define the operator ∇ ("called "del") to be

$$\nabla = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

Example Consider

$$F = F(x,y) = 3x^2y - 2y^3 - 5x^4y^2 + 6x^2$$

Then

$$\nabla F = \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}$$

$$= \left(6xy - 20x^3y^2 + 12x\right) + i\left(3x^2 - 6y^2 - 10x^4y\right)$$

Example

Consider

$$A = A(x,y) = 2xy - ix^2y^3$$

then

$$\nabla A = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(2xy - ix^2y^3\right)$$

$$= \frac{\partial}{\partial x}\left(2xy - ix^2y^3\right) + i\frac{\partial}{\partial y}\left(2xy - ix^2y^3\right)$$

$$= 2y - 2ixy^3 + i\left(2x - 3ix^2y^2\right)$$

$$= \left(2y + 3x^2y^2\right) + i\left(-2xy^3 + 2x\right)$$

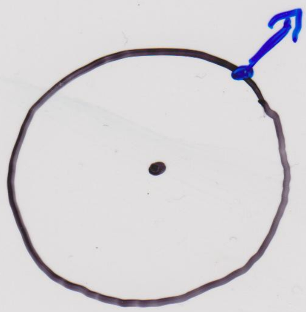
<u>Defn</u> We define the <u>gradient</u> of $F(x,y)$ or $A(x,y)$ to be

$$\text{grad } F = \nabla F$$

$$\text{grad } A = \nabla A.$$

<u>Interpretation</u> If $F(x,y) = c$ then ∇F represents a vector normal to the curve.

Example 1 $x^2 + y^2 = 2$



Find a normal to this curve at the point $(1,1)$.

$F(x,y) = x^2 + y^2$

$$\nabla F = \frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = 2x + i2y$$

At $(1,1)$ we have

$$\nabla F = 2 + 2i$$

Example Let C be the curve

$$3x^2y - 2y^3 = 5x^4y^2 - 6x^2.$$

Find a unit normal to C at $(1,-1)$.

$F = F(x,y) = 3x^2y - 2y^3 - 5x^4y^2 + 6x^2$

∇F is calculated above.

At $(1,-1)$ we have

$\nabla F = -14 + 7i$

Unit normal is

$\frac{1}{|\nabla F|} \nabla F$ = etc.

Divergence

Consider $A(x,y) = P(x,y) + i\,Q(x,y)$

Defn $\text{div} A = \nabla \cdot A$

$$= \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) \cdot (P + iQ)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

Example $A(x,y) = 2xy - ix^2y^3$

$$\operatorname{div} A = \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2y^3)$$

$$= 2y - 3x^2y^2 .$$

Curl (for $A = P + iQ$)

Defn $\operatorname{Curl} A = \nabla \times A$

$$= \begin{vmatrix} I & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ P & Q & 0 \end{vmatrix}$$

$$= \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) .$$

Observations

1) $\operatorname{grad}(A_1 + A_2) = \operatorname{grad} A_1 + \operatorname{grad} A_2$

2) $\operatorname{div}(A_1 + A_2) = \operatorname{div} A_1 + \operatorname{div} A_2$

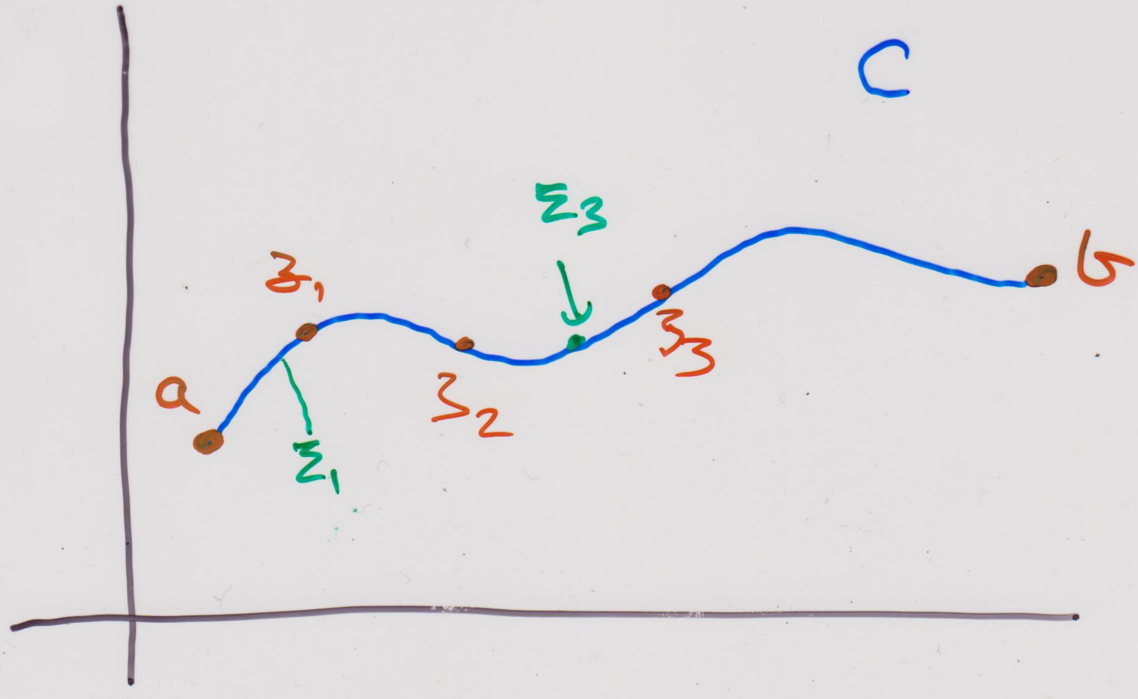
3) $\operatorname{Curl}(A_1 + A_2) = \operatorname{Curl} A_1 + \operatorname{Curl} A_2$

4) $\operatorname{grad}(A_1 A_2) =$

$$A_1 \operatorname{grad} A_2 + A_2 \operatorname{grad} A_1$$

Chapter 4 Integration

Let $f(z)$ be continuous at all points on a curve C in $\mathbb{C}$.



Choose $n-1$ points $z_1, z_2, \ldots, z_{n-1}$ on C (with $z_0 = a$, $z_n = b$)

Choose ξ_i on the curve between z_{i-1} and z_i.

Form the sum

$$S_n = f(\xi_1)(z_1 - a_1) + f(\xi_2)(z_2 - z_1) + \cdots + f(\xi_n)(b - z_{n-1}).$$

Introduce

$$\Delta z_i = z_i - z_{i-1}.$$

Then

$$S_n = \sum_{i=1}^{n} f(\xi_i)\Delta z_i.$$

if we let $n \to \infty$ in such a way that each Δz_i tends to 0.

Then

$$\lim_{n \to \infty} S_n$$

exists and is denoted by

$$\int_C f(z)\,dz \quad \text{or} \quad \int_a^b f(z)\,dz.$$

We call this the (complex) line integral of $f(z)$ along C.