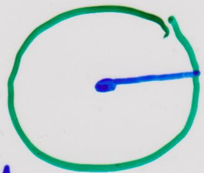
Singularities (Continued)

Branch points of $f(z)$ are singular points. If z_0 is a branch point then for any sufficiently small $\delta > 0$ the disk

$$|z_0 - z| < \delta$$



contains some choice of branch line. So $f(z)$ is not continuous at any point of the branch line, and hence not analytic.

So branch points are non-isolated singularities.

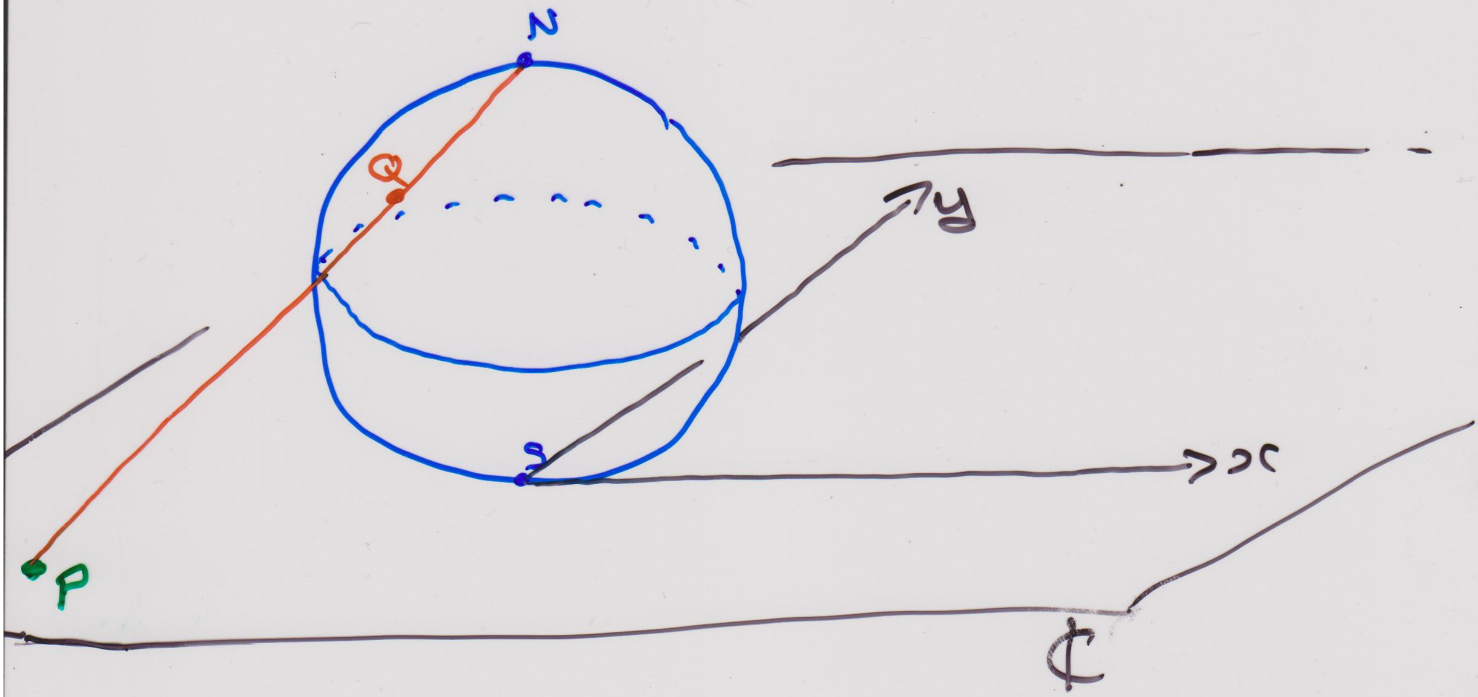
Stereographic Projection

Consider the sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$$

We can define a mapping

$$\mathbb{C} \longrightarrow S^2$$



which send a point $P \in \mathbb{C}$ to the point Q where the ray PN intersects the sphere.

Any point $Q \neq N$ on the sphere corresponds to a $P \in \mathbb{C}$.

We say that N corresponds to the "point at infinity of $\mathbb{C}$". We denote this point by ∞.

Singularities at infinity

We say that $f(z)$ has a singularity at ∞ if $f(\frac{1}{w})$ has a singularity at $w=0$.

<u>Example</u> Consider

$$f(z) = \frac{z^8 + z^4 + 2}{(z-1)^3(3z+2)^2}.$$

Does this function have a singularity at ∞ ?

<u>Sol^n</u>

$$f\left(\frac{1}{w}\right) = \frac{\left(\frac{1}{w}\right)^8 + \left(\frac{1}{w}\right)^4 + 2}{\left(\frac{1}{w}-1\right)^3\left(\frac{3}{w}+2\right)^2}$$

$$= \frac{1 + w^4 + 2w^8}{w^3 (1-w)^3 (3+2w)^2}.$$

Since

$$\lim_{w \to 0} (w-0)^3 f\left(\frac{1}{w}\right) = \frac{1}{3}$$

we see that $w=0$ is a pole of order 3 of $f\left(\frac{1}{w}\right)$.

So we say that $f(z)$ has a pole of order 3 at ∞.

Orthogonal Families of Curves

Let $w = f(z) = u(x,y) + i\,v(x,y)$

be analytic and $f'(z) \neq 0$.

Choose two real constants α, β

and consider the curves

$$u(x,y) = \alpha \quad \text{and} \quad v(x,y) = \beta .$$

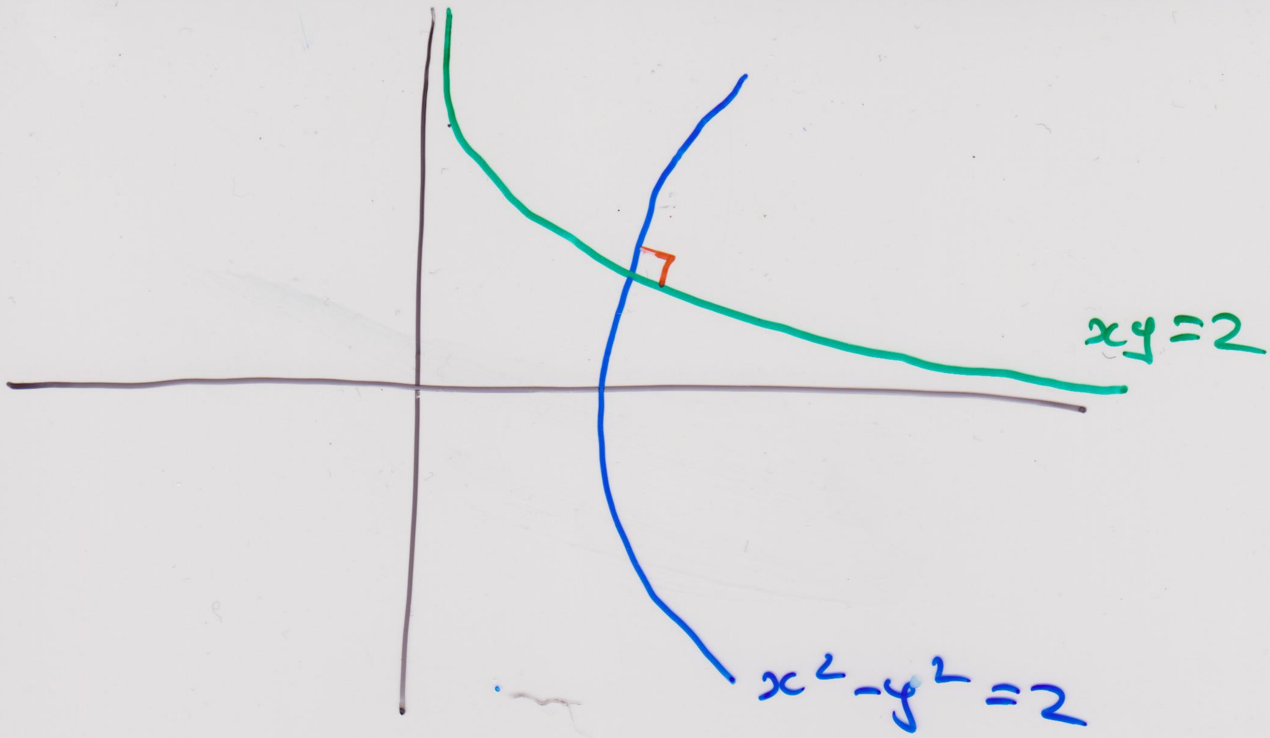
Example $f(z) = z^2, \quad \alpha = 2, \ \beta = 4.$

$$z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + i\,\underbrace{2xy}_{v(x,y)}$$

so we have curves

$$x^2 - y^2 = 2$$

$$2xy = 4$$



<u>Proposition</u> If $f(z) = u + iv$ is analytic and $f'(z) \neq 0$, then for any real constants α, β the curves

$$u(x,y) = \alpha$$
$$v(x,y) = \beta$$

intersect at right angles.