Cauchy-Riemann Equations (Continued)

Suppose $f(z) = u + iv$ is analytic on R. Then, for $z \in R$ the limit

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}$$

exists.

<u>Case 1</u> Suppose $\Delta y = 0$. Then

$$\text{limit} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} \right\} + i \left\{ \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right\}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad (1)$$

exists.

Case 2 Suppose $\Delta x = 0$. Then

$$\text{limit} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad (2)$$

For $f(z)$ to be analytic we need $(1) = (2)$.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Hence

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}. \end{aligned}\right\} \text{ Cauchy-Riemann equations.}$$

Theorem If $f(z) = u(x,y) + i v(x,y)$ is analytic on a region R then the Cauchy-Riemann equations hold in R.

Remember: Class Test next Monday 4th Feb.

Theorem Let $f(z)$, $g(z)$ be analytic in a region containing the point z_0. Suppose that $f(z_0) = g(z_0) = 0$ but $g'(z_0) \neq 0$. Then

$$\boxed{\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}}$$

Problem Evaluate

$$\lim_{z \to i} \frac{z^{10}+1}{z^6+1}$$

Set $f(z) = z^{10}+1$, $g(z) = z^6+1$.

Note $f(i) = 0$, $g(i) = 0$.

$f'(z) = 10z^9$

$g'(z) = 6z^5$, $g'(i) \neq 0$.

Hence

$$\lim_{z \to i} \frac{z^{10}+1}{z^{6}+1} = \frac{10(i)^9}{6(i)^5} = \frac{5}{3}$$

HARMONIC FUNCTIONS

Defn A function $u(x,y) : \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

Theorem (from chapter V)

If $f(z)$ is analytic on a disc R then all higher derivatives $f'(z), f''(z), f'''(z), \cdots$ exist in R.

<u>Corollary</u> Suppose that $f(z) = u + iv$ is analytic in a disk R. Then both u and v are harmonic.

<u>Proof</u> $f(z)$ analytic $\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

i.e. u is harmonic.

Exercise: v is harmonic.

Singular points

Defn A point at which $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$.

Defn A singular point $z = z_0$ of $f(z)$ is isolated if we can find $\delta > 0$ such that the disk

$$|z - z_0| < \delta$$

contains no other singular point of $f(z)$.

Defn If z_0 is an isolated singular point of $f(z)$ for which we can find an integer n

such that

$$\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0$$

then z_0 is a pole of order n.

Example Find and describe the singularities of

$$f(z) = \frac{z}{(z^2+4)^2}.$$

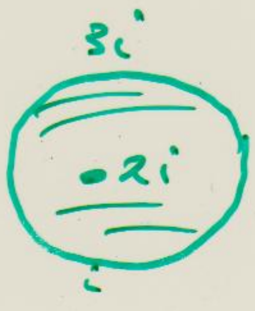
Sol^n

$$f(z) = \frac{z}{(z+2i)^2 (z-2i)^2}$$

Since

$$\lim_{z \to 2i} (z-2i)^2 f(z) = \lim_{z \to 2i} \frac{z}{(z+2i)^2} = \frac{1}{8i} \neq 0$$

and since $z = 2i$ is an isolated singularity of $f(z)$



we conclude that $z = 2i$ is a pole of $f(z)$ of order 2.

Exercise: $z = -2i$ is also a pole of order 2.