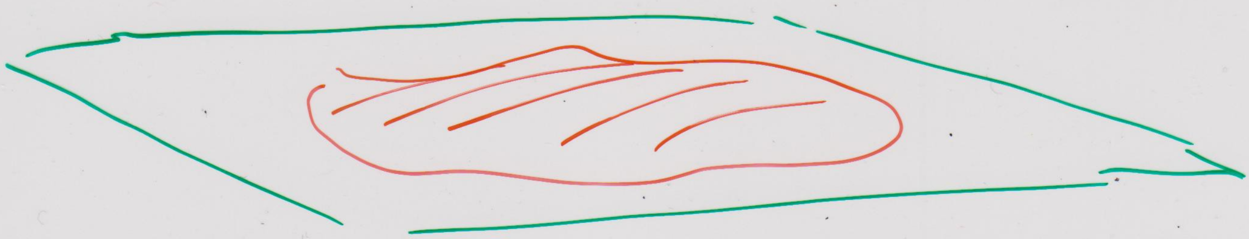
Chapter 3: Derivatives

we write

$$\lim_{z \to i} \frac{z^{10}+1}{z^{6}+1} = \frac{5}{3}$$

to mean that for all z close to i, but distinct from i, the expression $\frac{z^{10}+1}{z^6+1}$ has value close to $\frac{5}{3}$.

Let $f(z)$ be a single valued function defined in some region $R \subseteq \mathbb{C}$



Defn The derivative of $f(z)$ is

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

provided this limit exists.

Example For $f(z) = z^2 - 2z$ find $f'(z)$.

Soln

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - 2(z+\Delta z) - z^2 + 2z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^2} + 2z\Delta z + (\Delta z)^2 - \cancel{2z} - 2\Delta z - \cancel{z^2} + \cancel{2z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{2z + \Delta z - 2}{1}$$

$$= 2z - 2.$$

Defn If $f'(z)$ exists for all $z \in R$ then we say that $f(z)$ is *analytic* or *holomorphic* in R.

Defn We say that $f(z)$ is *analytic* at a point $z_0 \in \mathbb{C}$ if, for some $0 < \delta \in \mathbb{R}$, $f(z)$ is analytic in the region $|z - z_0| < \delta$.

Example $f(z) = z^2 - 2$ is analytic in $\mathbb{C}$.

Example $f(z) = \bar{z}$
This is *not* analytic at any point.

To see this, consider

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \overline{z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x+iy+\Delta x+i\Delta y} - \overline{x+iy}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x+\Delta x - i(y+\Delta y) - x + iy}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \qquad (*)$$

Case 1

suppose $\Delta x = 0$.

$$(*) = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

Case 2

suppose $\Delta y = 0$.

$$(*) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

Hence the limit does not exist and $f(z)$ is not analytic at any $z \in \mathbb{C}$.

Cauchy-Riemann Equations

Any function $f(z)$ can be written in the form

$$\boxed{f(z) = u(x,y) + i\,v(x,y)}$$

Example Consider $f(z) = z^2 - 2z$.

$$f(x+iy) = (x+iy)^2 - 2(x+iy)$$

$$= x^2 - y^2 + 2ixy - 2x - 2iy$$

$$= \underbrace{(x^2 - y^2 - 2x)}_{u(x,y)} + i\,\underbrace{(2xy - 2y)}_{v(x,y).}$$

Theorem If $f(z) = u(x,y) + i v(x,y)$ is analytic in a region $R \subseteq \mathbb{C}$ then the following **Cauchy-Riemann** equations hold:

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}}$$

Example verify the Cauchy-Riemann equations for

$$f(z) = z^2 - 2z .$$

sol^n

$$\frac{\partial u}{\partial x} = 2x - 2 = \frac{\partial v}{\partial y} = 2x - 2$$

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x} = -2y$$