Defn If $z = e^{w}$ then we write

$$w = \ln(z).$$

If $z = r(\cos\theta + i\sin\theta)$

then

$$\ln(z) = \ln(r) + i(\theta + 2k\pi)$$

$$k = 0, \pm 1, \pm 2, \dots .$$

The principal value or principal branch of $\ln(z)$ could be defined to be

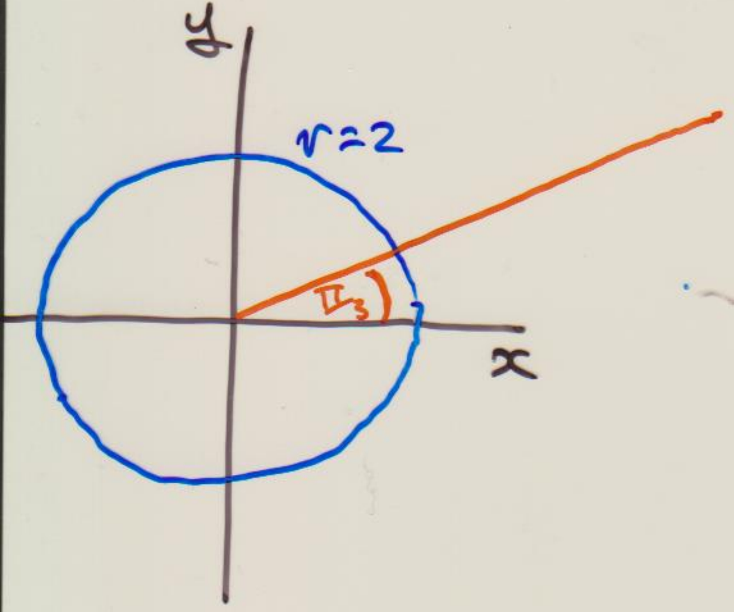
$$\ln(r) + i\theta$$

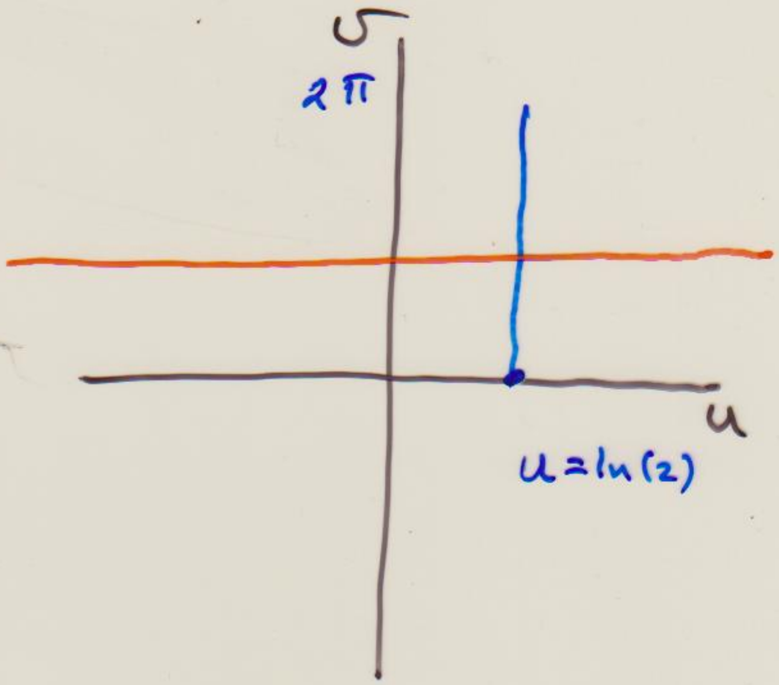
where $0 \le \theta < 2\pi$.

Remark We can regard this principal value of the logarithm as a single valued function

$$\mathbb{C}^* = \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$$

$$z = x + iy \longmapsto \ln(z) = u + iv$$





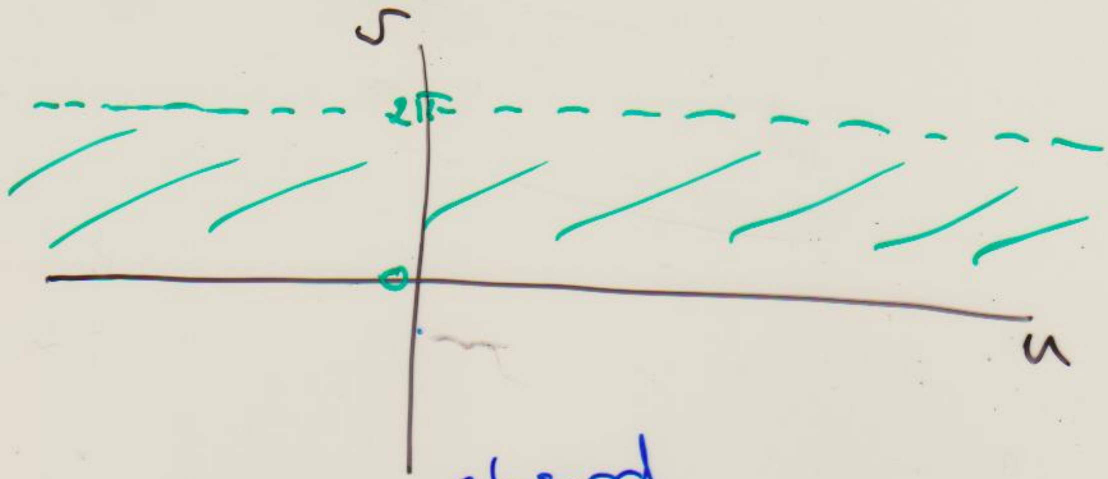
z-plane
or
x-y-plane

w-plane
or
u-v-plane.

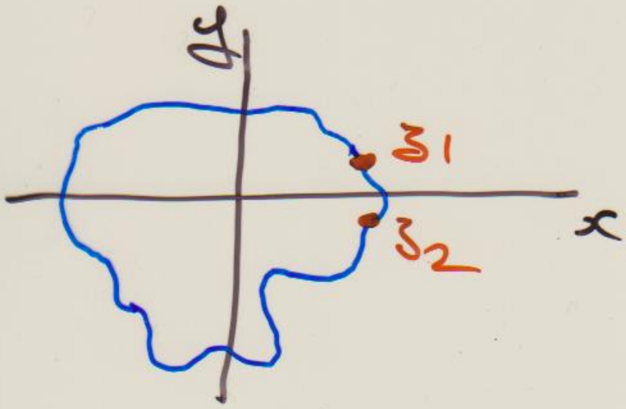
Circles about origin $\longmapsto$ vertical straight line segments

rays from origin $\longmapsto$ horizontal lines

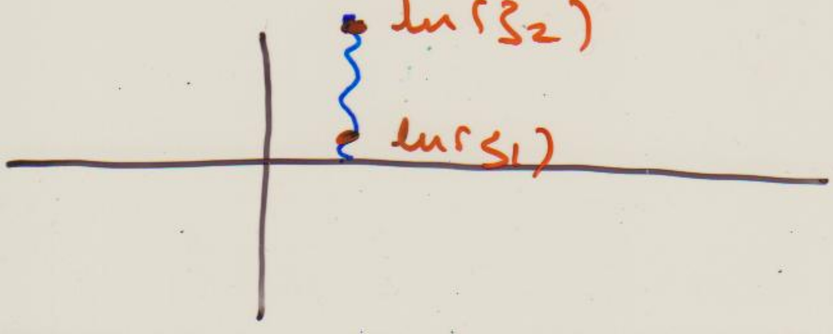
Note that our principal value of $\ln(z)$ maps $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ into the strip



Consider any closed circuit <u>about the origin</u> in the z-plane.



our chosen principal value of $\ln(z)$ maps this circuit to a curve



The circuit is ~~not~~ mapped continuously because nearby points do not necessarily get sent to nearby points.

Mathematics prefers continuous functions!

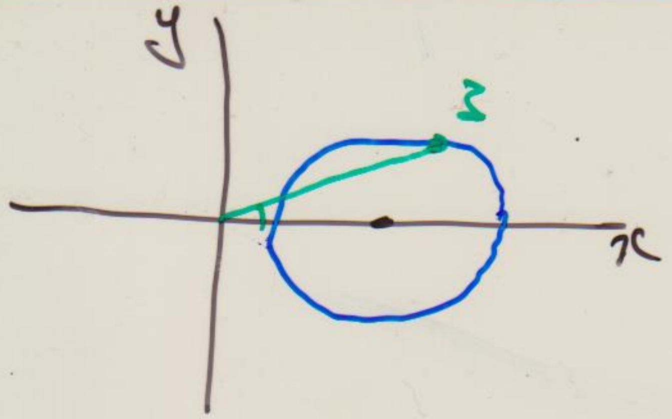
Branch Points, Branch lines and Riemann Surfaces

Consider the function

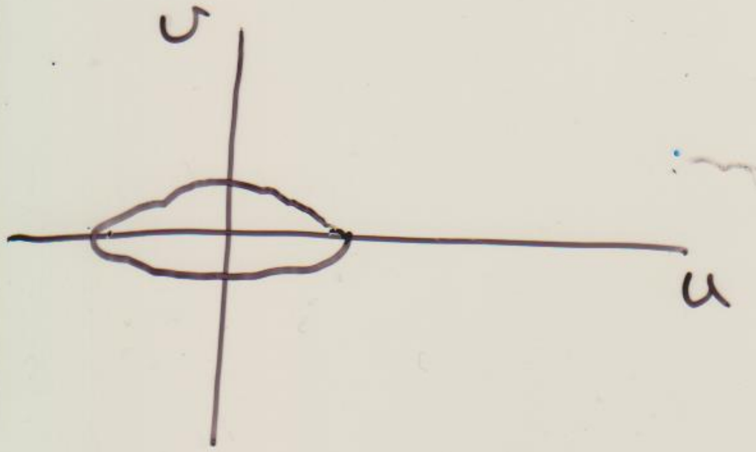
$$w = \ln(z).$$

Any circuit about the origin is mapped discontinuously by any/ choice of principal value.

However, any circuit not about the origin gets mapped continuously using a suitable choice of principal value.



gets mapped to

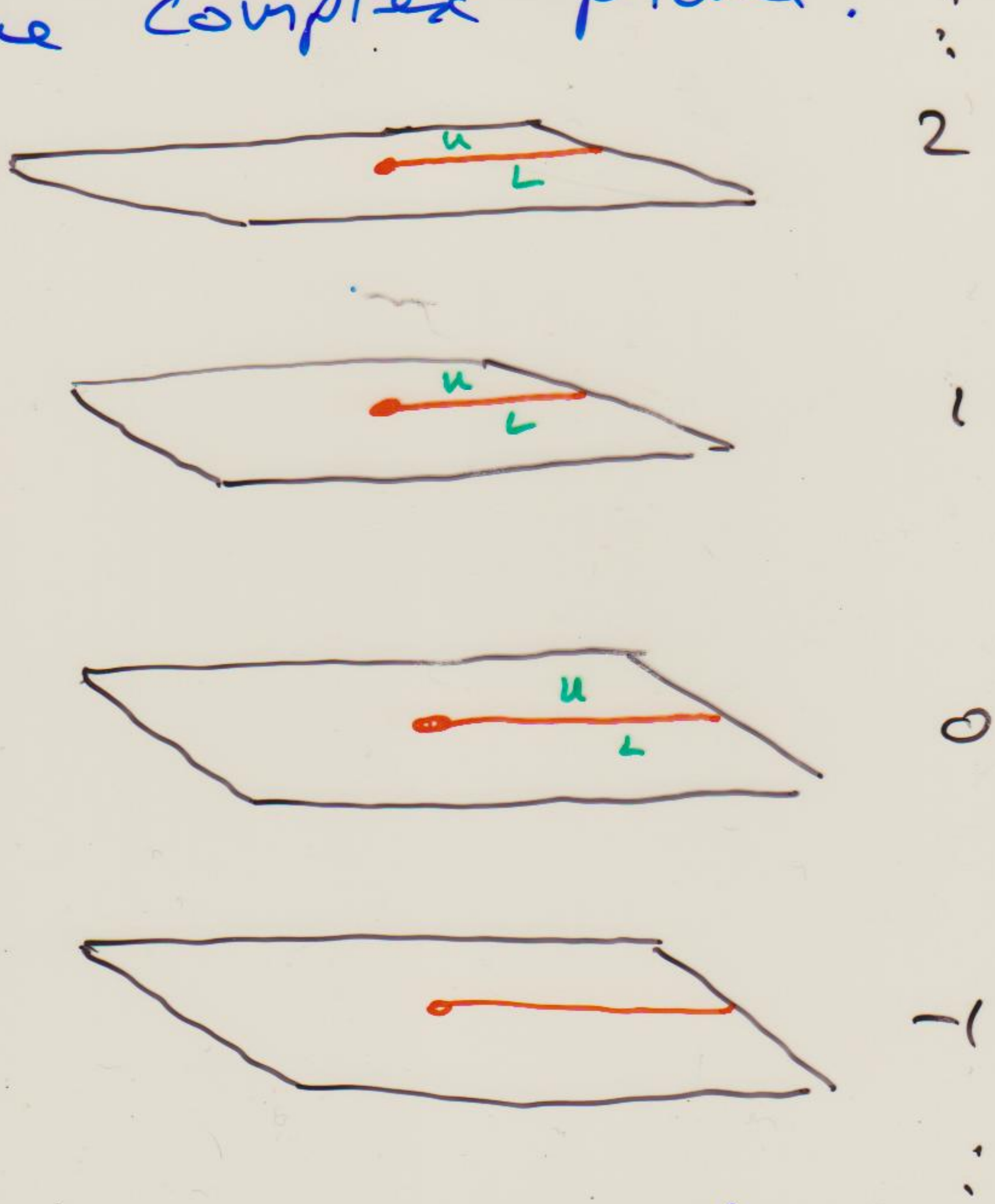


chosing principle value $-\pi < \theta \leq \pi$

We thus say that the origin $z=0$ is a <u>branch point</u> for $w = \ln(z)$.

Any ray from this branch point is called a <u>branch line</u>.

To view $w = \ln(z)$ as a continuous function we take countably many copies of the complex plane.



cut each copy along its branch line.

We then glue the upper face of the branch line of the n^{th} copy of $\mathbb{C}$ to the lower face of the branch line of the $(n+1)^{st}$ copy of $\mathbb{C}$.

The result is a "Riemann surface" S which is:

1) locally like $\mathbb{C}$;
2) contains no closed circuit around the origin.
3) allows us to consider a continuous and single valued function $S \to \mathbb{C}$, $z \mapsto \ln(z)$.