Recall

$$z = x + iy$$

$$\bar{z} = x - iy$$

Proposition

$$\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$$

and

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

Proof exercise

Corollary Let $w \in \mathbb{C}$ be a zero of the polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

where the coefficients $a_0, a_1, \ldots, a_n$ are all real numbers. Then $\bar{w}$ is also a zero of $p(z)$.

Proof Suppose $p(w) = 0$.

Then

$$P(\bar{w}) = a_0 + a_1\bar{w} + a_2\bar{w}^2 + \cdots + a_n\bar{w}^n$$

$$= \bar{a_0} + \bar{a_1}\bar{w} + \bar{a_2}\bar{w}^2 + \cdots + \bar{a_n}\bar{w}^n$$

$$= \bar{a_0} + \overline{a_1 w} + \overline{a_2 w^2} + \cdots + \overline{a_n w^n}$$

$$= \overline{a_0 + a_1 w + a_2 w^2 + \cdots + a_n w^n}$$

$$= \overline{P(w)}$$

$$= \bar{0}$$

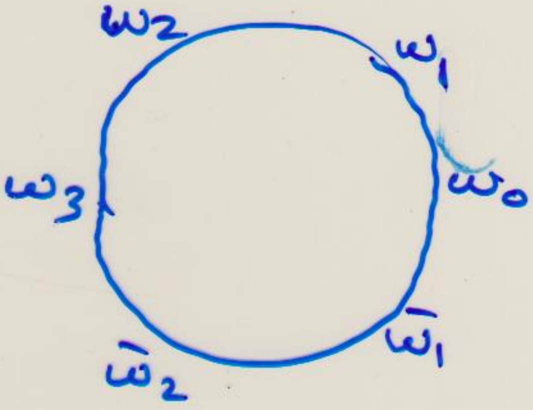
$$= 0.$$

$\blacksquare$

<u>Problem</u> Factorize $x^6 - 1$ as a product of real linear and quadratic factors.

<u>Sol^n</u> Let $w_k = \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}$ for $k = 0, 1, \cdots, 5$. These are the 6^{th} roots of unity.

$$x^6-1=(x-\omega_0)(x-\omega_1)(x-\omega_2)(x-\omega_3)$$
$$\times(x-\omega_4)(x-\omega_5)$$



$$=(x-1)(x+1)$$
$$\times\left(x^2-(\omega_1+\bar{\omega}_1)x+\omega_1\bar{\omega}_1\right)$$
$$\times\left(x^2-(\omega_2+\bar{\omega}_2)+\omega_2\bar{\omega}_2\right)$$

$$x^6-1=(x-1)(x+1)\left(x^2+2\cos\left(\tfrac{\pi}{3}\right)x+1\right)$$
$$\times\left(x^2+2\cos\left(\tfrac{2\pi}{3}\right)x+1\right).$$

Chapter Two: Functions

A single valued function of a complex variable is a rule

$$f:\mathbb{C}\longrightarrow\mathbb{C},\quad z\longmapsto f(z)$$

assigning $f(z)\in\mathbb{C}$ to $z\in\mathbb{C}$.

Example

$$f(z) = z^2$$

or we might write

$$w = z^2$$

is a single valued function.

Example $f(z) = z^{\frac{1}{6}}$ is **not** a single valued function. For instance, $f(i)$ has six possible values.

We often say that $f(z) = z^{\frac{1}{6}}$ or $w = z^{\frac{1}{6}}$ is a multi valued function.

Elementary Functions

$$w = e^z = e^x(\cos x + i \sin y)$$

where $z = x + iy$

$$w = \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$w = \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

Exercises

$$\sin(iz) = i\sinh(z)$$

$$\sin^2(z) + \cos^2(z) = 1$$

$$\cosh^2(z) - \sinh^2(z) = 1$$

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Defn if $z = e^{w}$ then we write

$$w = \ln(z)$$

So if

$$z = r(\cos\theta + i\sin\theta)$$

then

$$\ln(z) = \ln(r) + i(\theta + 2k\pi)$$

$$k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

[This is because

$$e^{\ln(r) + i(\theta + 2k\pi)}$$

$$= e^{\ln(r)} e^{i\theta} e^{i2k\pi}$$

$$= r(\cos\theta + i\sin\theta) . 1$$

$$= z \quad]$$

So

$$w = \ln(z)$$

is a multivalued function.

For each z there are infinitely many values of $\ln(z)$.