Test next Monday

chapters V, VI, VII

Tutorial: ~~T~~ Wednesday 17th April
11 am

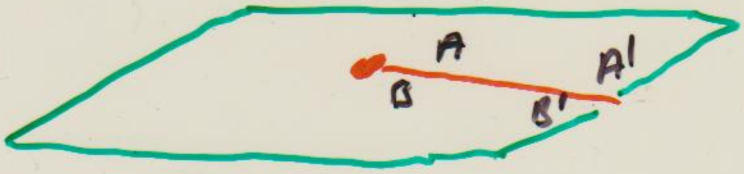
Problem

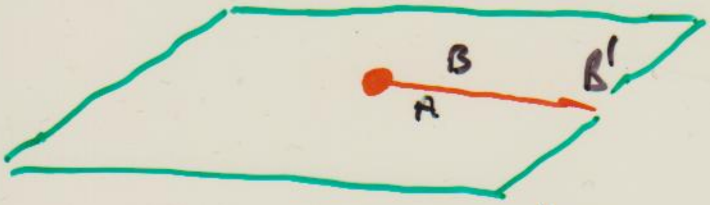
Evaluate $\displaystyle\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

Soln Consider

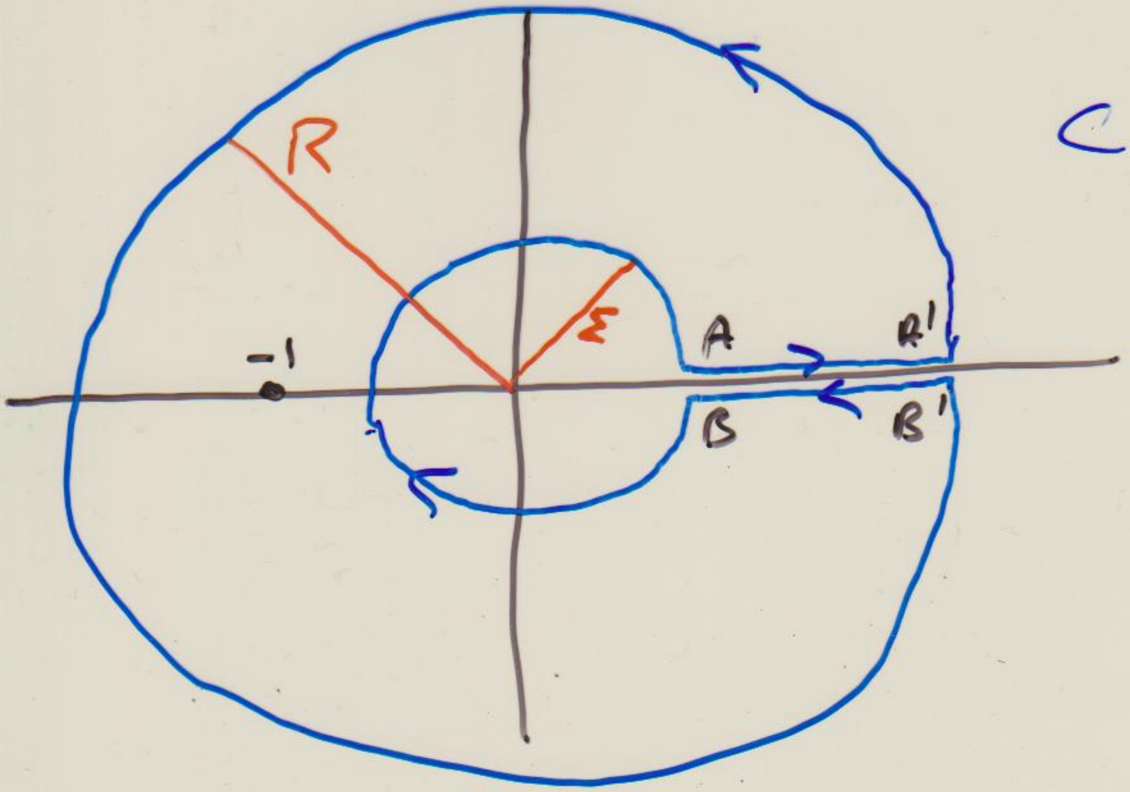
$$f(z) = \frac{1}{(1+z)\sqrt{z}}.$$

Now $z=0$ is a branch point.
The positive real axis is a branch line. A Riemann surface involves two copies of $\mathbb{C}$.





Consider the following curve C:



Now $f(z) = \dfrac{1}{(1+z)\sqrt{z}}$

has just one pole <u>inside</u> C.

This pole is at $z = -1$ and has order 1

Residue of $f(z)$ at $z = -1$ is

$$a_{-1} = \lim_{z \to -1} (z+1) \frac{1}{(z+1)\sqrt{z}} = -i$$

Residue Theorem says

$$\oint_C \frac{dz}{(1+z)\sqrt{z}} = 2\pi i(-i) = 2\pi$$

$z = Re^{i\theta}$
$dz = iRe^{i\theta}\, d\theta$

So

$$2\pi = \int_{\varepsilon}^{R} \frac{dx}{(1+x)\sqrt{x}} + \cancel{\int_0^{2\pi} \frac{iRe^{i\theta}\, d\theta}{(1+Re^{i\theta})\sqrt{Re^{i\theta}}}} \rightarrow 0 \text{ as } R \to \infty$$

$$+ \int_R^{\varepsilon} \frac{dx}{(1+xe^{2\pi i})\sqrt{xe^{2\pi i}}} + \cancel{\int_{2\pi}^{0} \frac{i\varepsilon e^{i\theta}\, d\theta}{(1+\varepsilon e^{i\theta})\sqrt{\varepsilon e^{i\theta}}}} \rightarrow 0 \text{ as } \varepsilon \to 0$$

As $\varepsilon \to 0$, $R \to \infty$

$$2\pi = \int_0^\infty \left(\frac{1}{(1+x)\sqrt{x}} - \frac{1}{(1+x)\sqrt{x}\, e^{\pi i}} \right) dx$$

$$2\pi = \int_0^\infty \frac{1}{(1+x)\sqrt{x}} \left(1 - \frac{1}{e^{\pi i}} \right) dx$$

$$2\pi = \left(1 - \frac{1}{-1} \right) \int_0^\infty \frac{1}{(1+x)\sqrt{x}}\, dx$$

$$\pi = \int_0^\infty \frac{1}{(1+x)\sqrt{x}}\, dx \quad .$$