Problem

Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^4+1}$

Sol^n

Last lecture we showed

$$\int_0^\infty \frac{dx}{x^4+1} = \pi i\,(a_{-1} + b_{-1})$$

where a_{-1}, b_{-1} are the residues of $f(z) = \dfrac{1}{z^4+1}$ at $e^{\frac{\pi i}{4}}$, $e^{\frac{3\pi i}{4}}$.

Recall :

$$a_{-1} = \lim_{z\to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left\{ (z-a)^k f(z) \right\}$$

For $a = e^{\frac{\pi i}{4}}$, $k=1$ L'Hôpital's rule

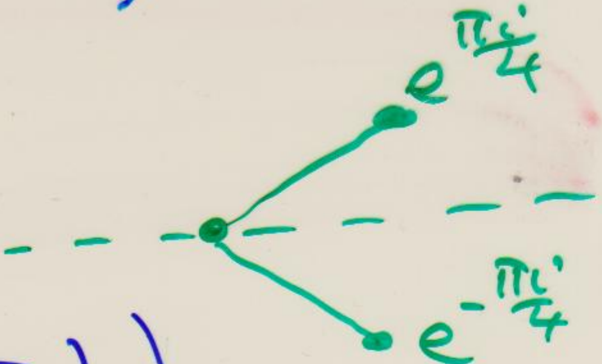
$$a_{-1} = \lim_{z\to e^{\frac{\pi i}{4}}} \frac{\left(z - e^{\frac{\pi i}{4}}\right)}{(z^4+1)} =$$

$$a_{-1} = \lim_{z \to e^{\frac{\pi i}{4}}} \frac{1}{4z^3} = \frac{1}{4} e^{-\frac{3\pi i}{4}}$$

$$b_{-1} = \lim_{z \to e^{\frac{3\pi i}{4}}} \frac{1}{4z^3} = \frac{1}{4} e^{-\frac{\pi i}{4}}$$

$$(a_{-1} + b_{-1})\pi i = \frac{\pi i}{4}\left(e^{-\frac{\pi i}{4}} + e^{-\frac{3\pi i}{4}}\right)$$

$$= \frac{\pi}{4}\left(e^{\frac{\pi i}{4}} + e^{-\frac{\pi i}{4}}\right)$$

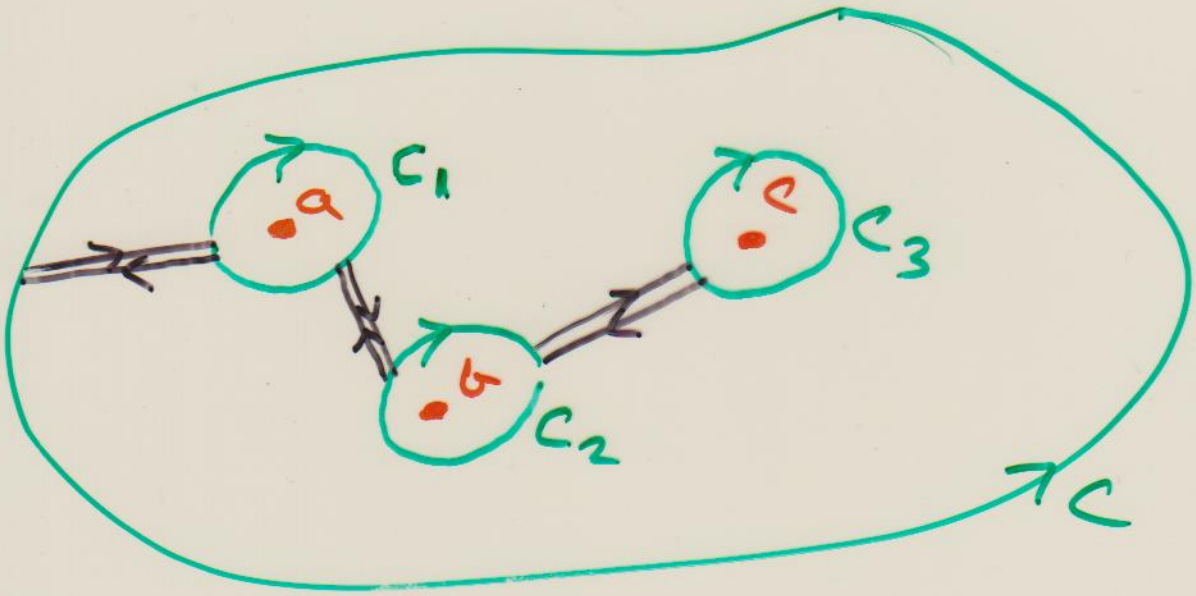


$$= \frac{\pi}{4}\left(\cos\left(\frac{\pi}{4}\right) + \cos\left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi}{2\sqrt{2}} . = \int_0^\infty \frac{1}{x^4+1}\,dx$$

Proof of Residue Theorem

Let $f(z)$ be analytic inside and on a simple closed curve C, except at singularities $a, b, c, \ldots$ in C



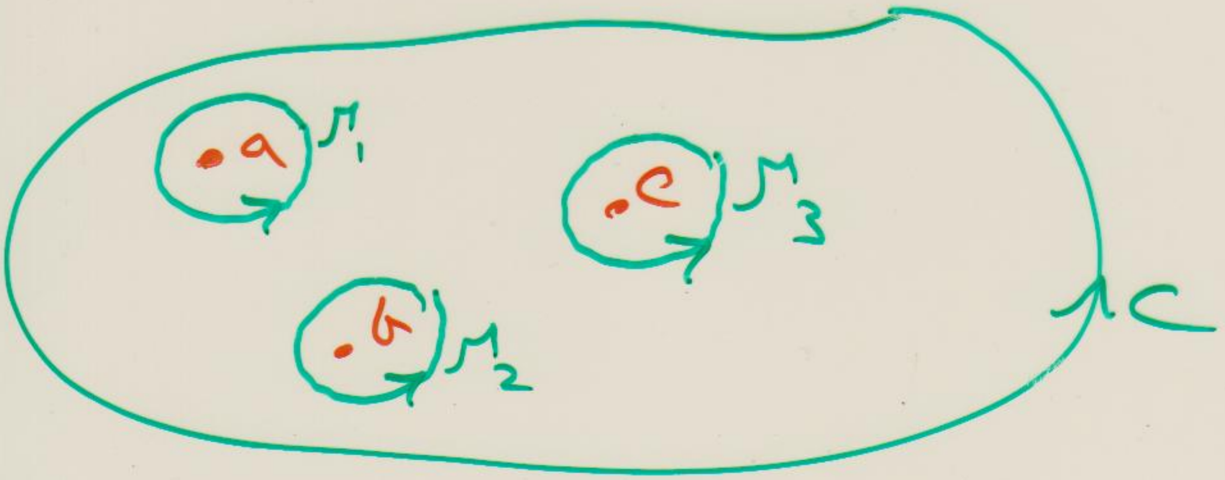
$$\oint_C f(z)\,dz + \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \int_{c_3} f(z)\,dz = 0$$

by Cauchy's Theorem, since:

1) $f(z)$ is analytic on the interior of the simple closed curve formed from $C, C_1, C_2, C_3, \ldots$ and the black bridges

2) the integral over all the bridges is zero.

Thus



$$\oint_C f(z)\,dz = \oint_{\Gamma_1} f(z)\,dz + \oint_{\Gamma_2} f(z)\,dz + \oint_{\Gamma_3} f(z)\,dx \ldots$$

Laurent's Theorem (whose proof is analogous to that of Taylor's Theorem) says

$$a_{-1} = \frac{1}{2\pi i} \oint_{\Gamma_1} f(z)\, dz \,,$$

$$b_{-1} = \frac{1}{2\pi i} \oint_{\Gamma_2} f(z)\, dz \qquad \text{etc.}$$

So

$$\oint_C f(z)\, dz = 2\pi i \,(a_{-1} + b_{-1} + c_{-1} + \cdots)$$

$\boxed{\text{ML}}$