Recall

The Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

of $f(z) = \dfrac{z}{(z+1)(z+2)}$ about $a=-2$

is

$$\frac{z}{(z+1)(z+2)} = \frac{2}{(z+2)} - 1 + (z+2) + (z+2)^2 + \cdots$$

is convergent for $0 < |z+2| < 1$.

The coefficient a_{-1} is called the residue of $f(z)$ about a.

So the residue of $f(z) = \dfrac{z}{(z+1)(z+2)}$ about $a=-2$ is $a_{-1} = 2$.

Proposition If $z = a$ is a pole of order k of $f(z)$ then

$$a_{-1} = \lim_{z \to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z-a)^k f(z) \right].$$

Illustration

Consider $f(z) = \dfrac{z}{(z+1)(z+2)}$.

Find the residue at $z = -2$.

$z = -2$ is a pole of order 1

since

$$\lim_{z \to -2} (z+2)^1 f(z) = A \neq 0 \text{ exists}.$$

So

$$a_{-1} = \lim_{z \to -2} \frac{1}{0!} \frac{d^0}{dz^0} (z+2) f(z)$$

$$= \lim_{z \to -2} \frac{z}{(z+1)} = 2.$$

Proof of Proposition

If $f(z)$ has a pole a of order k then

$$\lim_{z \to a} (z-a)^k f(z) = A \neq 0 \quad \text{exists.}$$

Then the Laurent expansion about a has the form

$$f(z) = \frac{a_{-k}}{(z-a)^k} + \frac{a_{-k+1}}{(z-a)^{k-1}} + \cdots + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + \cdots$$

Multiplying both sides by $(z-a)^k$ gives

$$(z-a)^k f(z) = a_{-k} + a_{-k+1}(z-a) + a_{-k+2}(z-a)^2 + \cdots$$

so

$$\frac{d}{dz}\left\{(z-a)^k f(z)\right\} = a_{-k+1} + 2a_{-k+2}(z-a) + 3a_{-k+3}(z-a)^2 + \cdots$$

$$\frac{d^{k-1}}{dz^{k-1}}\left\{(z-a)^k f(z)\right\} = (k-1)!\,a_{-1} + \frac{k!}{2}(z-a) + \dots$$

In the limit as $z \to a$

$$\lim_{z\to a} \frac{d^{k-1}}{dz^{k-1}}\left\{(z-a)^k f(z)\right\} = (k-1)!\,a_{-1}$$

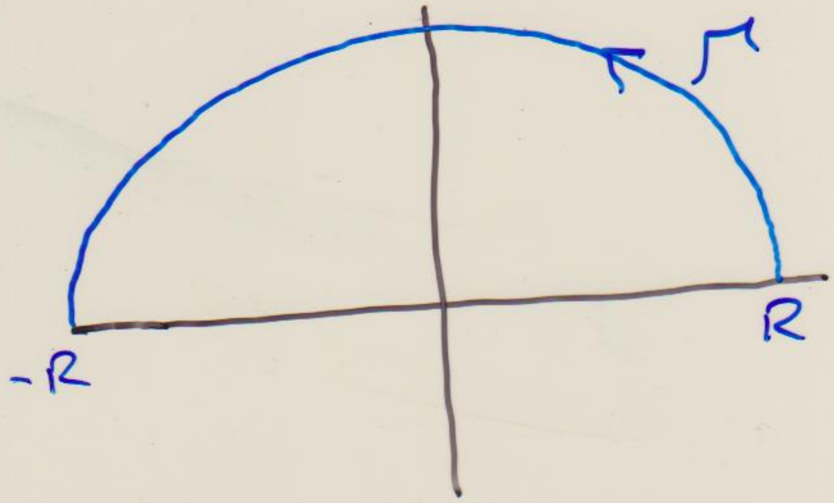
$\boxed{\text{W5}}$

Integrals of the type $\int_{-\infty}^{\infty} F(x)\,dx$.

<u>Prop</u> If $|F(z)| \leq \frac{M}{R^k}$ for $z = Re^{i\theta}$ where $k > 1$ and M are constants, then

$$\lim_{R\to\infty} \int_\Gamma F(z)\,dz = 0$$

for the semi-circular arc Γ



__Proof__

$$\left| \int_{\Gamma} F(z)\, dz \right| \leq \frac{M}{R^k} \cdot \pi R = \frac{\pi M}{R^{k-1}}$$

since $L = \pi R$ is the arc-length of Γ.

So

$$\lim_{R \to \infty} \left| \int_{\Gamma} F(z)\, dz \right| = 0 .$$

$\blacksquare$

Problem

Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^4+1}$

Sol^n

Consider $F(z) = \dfrac{1}{z^4+1}$

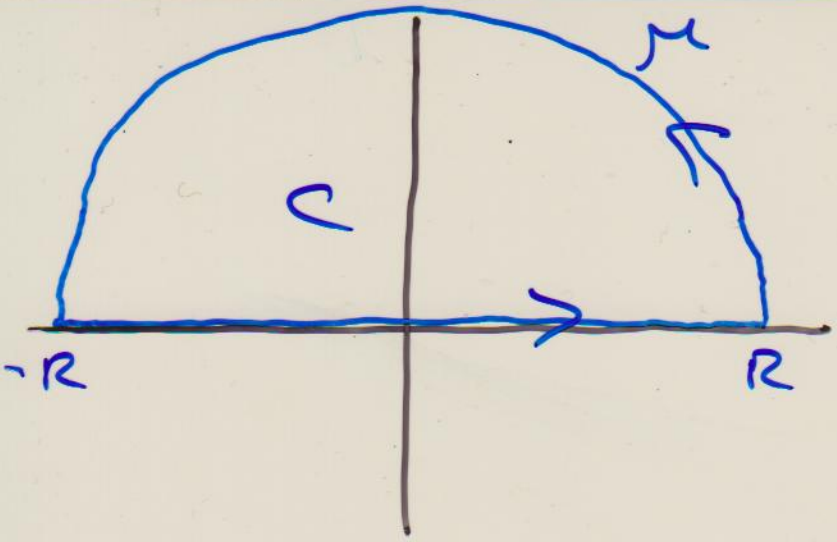
For $z = Re^{i\theta}$,

$$|F(z)| = \left| \frac{1}{R^4 e^{4i\theta}+1} \right|$$

$$\leqslant \frac{1}{|R^4 e^{4i\theta}|-1}$$

$$= \frac{1}{R^4-1}$$

$$\leq \frac{2}{R^4} \quad \text{for large } R.$$



$$\oint_C \frac{dz}{z^4+1} = \int_{-R}^{R} \frac{dz}{z^4+1} + \int_{\mu} \frac{dz}{z^4+1}$$

In the limit where C has infinite radius

$$\int_C \frac{dz}{z^4+1} = \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\int_0^{\infty} \frac{dx}{x^4+1}$$

Thus

$$\int_0^{\infty} \frac{dx}{x^4+1} = \frac{1}{2}\int_C \frac{dz}{z^4+1} = 2\pi i \text{ (sum of residues)}$$

The singularities of $\dfrac{1}{z^4+1}$

occur at

$$z = e^{\frac{\pi i}{4}},\ e^{\frac{3\pi i}{4}},\ e^{\frac{5\pi i}{4}},\ e^{\frac{7\pi i}{4}}.$$

Only $e^{\frac{\pi i}{4}}$ and $e^{3\pi i/4}$ occur in C.

Thus

$$\int_0^\infty \frac{dx}{x^4+1} = 2\pi i\,(a_{-1} + b_{-1})$$

where a_{-1}, b_{-1} are the residues of $f(z) = \dfrac{1}{z^4+1}$ at $z = e^{\pi i/4}$ and $z = e^{3\pi i/4}$.