Recall: For $f(z)$ analytic inside an on a simple closed curve C we have a Taylor expansion

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

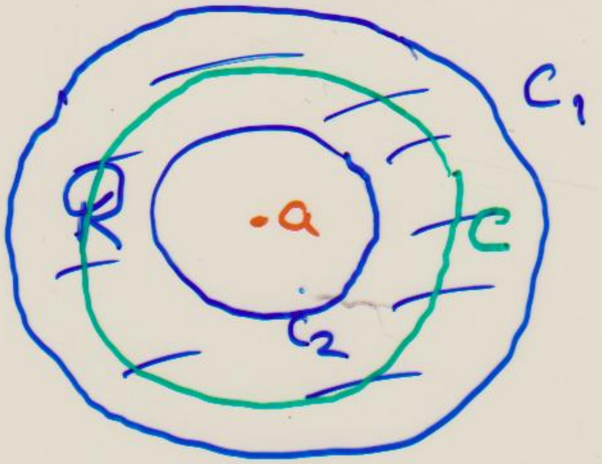
for z and a inside C.

Recall: proof used Cauchy's integral formulae

$$\frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-a)^{n+1}}\,dw$$

$n = 0, 1, 2, \ldots$

Laurent Theorem If $f(z)$ is analytic inside and on the boundary of the ring-



shaped region R bounded by two concentric circles centred at $z=a$, then for $z \in R$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \cdots$$

$$+ \frac{a_{-1}}{(z-a)} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-3}}{(z-a)^3} + \cdots$$

when

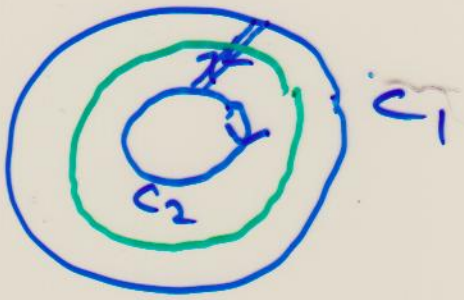
$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-a)^{n+1}}\, dw,$$

$$n = 0, \pm 1, \pm 2 \cdots$$

and C is a concentric circle between C_1 & C_2

Proof uses Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{w-z}\,dw$$



and involves letting the radii of C_1, C_2 tend to that of C.
(See book for details). ☐

Example Find the Laurent series for

$$f(z) = \frac{z}{(z+1)(z+2)}$$

about $z = -2$.

Sol^n Let $a = -2$

$u = z - a = z + 2$

$$\frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-1)\,u}$$

$$= \frac{2-u}{u} \cdot \frac{1}{1-u}$$

$$= \left(\frac{2}{u} - 1\right)\left(1 + u + u^2 + u^3 + \cdots\right)$$

$$= \frac{2}{u} + 1 + u + u^2 + u^3 + \cdots$$

$$= \frac{2}{(z+2)} + 1 + (z+2) + (z+2)^2 + (z+2)^3 + \cdots$$

This is convergent for

$|u| < 1$

or $|z+2| < 1$.

Chapter 7: Residue Theorem

Let $f(z)$ be single-valued and analytic inside and on a circle C except at the centre $z=a$ of C.

In the Laurent series expansion

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$$

we call

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

the residue of $f(z)$ at $z=a$.

The Residue Theorem

Let $f(z)$ be analytic inside and on a simple closed curve Γ

except a the isolated singularities $a, b, c, \dots$. Let $a_{-1}, b_{-1}, c_{-1}, \dots$ denote the residues of $f(z)$ at $a, b, c, \dots$.

Then

$$\oint_{\Gamma} f(z)\, dz = 2\pi i \left(a_{-1} + b_{-1} + c_{-1} \cdots \right)$$

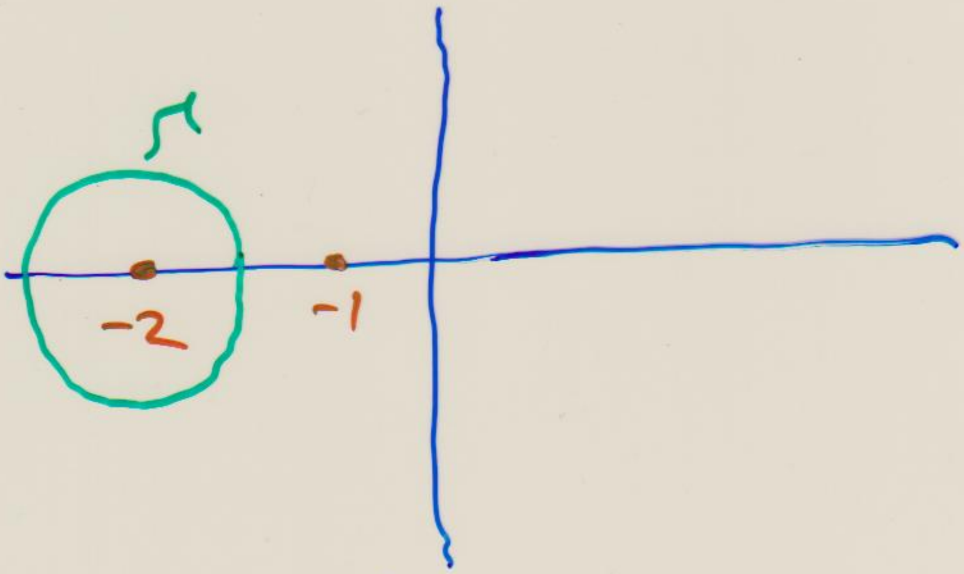
Example Calculate

$$\oint_{\Gamma} \frac{z}{(z+1)(z+2)}\, dz$$

where Γ is the circle

$|z+2| = \frac{1}{2}$.

Sol^n



The Residue Theorem says

$$\oint_{\Gamma} \frac{z}{(z+1)(z+2)}\, dz = a_{-1} =$$

where a_{-1} is the residue of

$f(z) = \dfrac{z}{(z+1)(z+2)}$ at $z = -2$.

well, about $a=-2$

$$f(z) = \frac{2}{(2+z)} + 1 + (z+2) + (z+2)^2 + \cdots$$

so $a_{-1} = 2$.

so

$$\int_\gamma \frac{z}{(z+1)(z+2)}\,dz = 2(2\pi i)$$
$$= 4\pi i$$