Test next Monday

Exam details

2 hours

4 questions

Attempt all 4 questions.

Each question has four parts and you should attempt three parts only.

Q1 i) chapters I & II
ii)
iii)
iv)

Q2 i) chapters III & IV
⋮
iv)

Q3 i) chapters V & VI
⋮
iv)

Q4 i) chapters VII plus one earlier chapter
⋮
iv)

All questions are based on the lecture slides and the exercise sheets. No unseen material.

More material is taken from the slides than from the sheets.

Typical questions:

- Define/explain a concept
- state/prove a result
- calculate/describe.

Ratio Test Suppose given a sequence $u_1, u_2, u_3, \ldots$ of complex numbers.

If $$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = L < 1$$

then $\sum_{n=1}^{\infty} u_n$ is convergent.

Example (Last lecture cont.)

$f(z) = \ln(1+z)$, $f(0) = 0$.

we saw

$$f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

n^{th} term is

$$u_n = (-1)^{n-1} \frac{z^n}{n}$$

now $$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{z^{n+1}\, n}{(n+1)\, z^n} \right| = \frac{n}{n+1} |z|$$

so $$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = L < 1 \text{ if } |z| < 1.$$

Thus the series converges

for $|z| < 1$

__Example__ Expand $f(z) = \sin(z)$ in a Taylor's series about $z = \frac{\pi}{4}$, and determine the region of convergence.

$f(z) = \sin(z)$ $\qquad f(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$f'(z) = \cos(z)$ $\qquad f'(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$f''(z) = -\sin(z)$ $\qquad f''(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$f'''(z) = -\cos(z)$ $\qquad f'''(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$

$f^{(iv)}(z) = \sin(z)$ $\qquad f^{(iv)}(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

For $a = \frac{\pi}{4}$

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

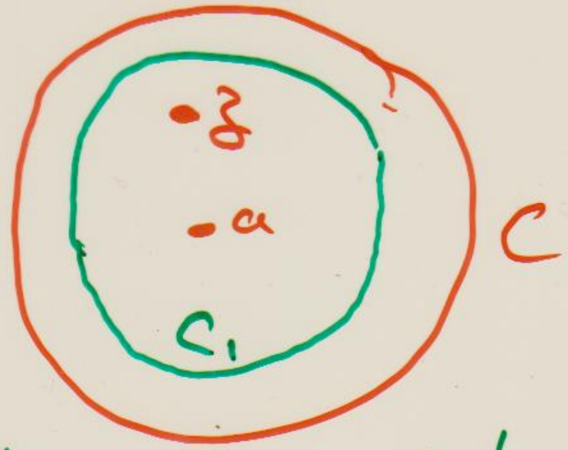
$$= \frac{\sqrt{2}}{2}\left\{1 + (z-\tfrac{\pi}{4}) - \frac{(z-\frac{\pi}{4})^2}{2} - \frac{(z-\frac{\pi}{4})^3}{3!} + \cdots\right\}$$

Since the singularity of $\sin(z)$ nearest to $a = \frac{\pi}{4}$ is at infinity, the region of convergence is $|z| < \infty$.

Partial proof of Taylor's Theorem.

Let $f(z)$ be analytic inside a circle C with centre a.

Let z be any point inside C.



Let C_1 be a circle centred at a enclosing z.

Cauchy's Integral formula

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\, dw \qquad (1)$$

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)}$$

$$= \frac{1}{w-a}\left\{\frac{1}{1-\frac{(z-a)}{(w-a)}}\right\}$$

$$= \frac{1}{w-a}\left\{1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}\right.$$

$$\left.+\left(\frac{z-a}{w-a}\right)^n\frac{1}{1-\frac{(z-a)}{(w-a)}}\right\}$$

Thus

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-a} + (z-a)\frac{f(w)}{(w-a)^2} + (z-a)^2\frac{f(w)}{(w-a)^3} + \cdots$$

Hence

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{w-a}\,dw + \frac{(z-a)}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^2}\,dw$$

$$+ \frac{(z-a)^2}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^3} + \cdots$$

Cauchy's Integral formulae:

$$\frac{f^{n}(a)}{n!} = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw$$

So

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a)$$
$$+ \frac{(z-a)^3}{3!} f'''(a) + \cdots$$

To complete the proof we need to check that this infinite series converges. (See Problem 22 in Chapter 6.)