Test next Monday, 4th March

Theorem Every polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

with $n \geq 1$, $a_n \neq 0$ has exactly n roots in $\mathbb{C}$.

Proof we know that $P(z)$ has at least one root $a \in \mathbb{C}$. i.e. $P(a) = 0$.

Now

$$P(z) = P(z) - P(a)$$

$$= (a_0 + a_1 z + \dots + a_n z^n) - (a_0 + a_1 a + \dots + a_n a^n)$$

$$= a_1(z-a) + a_2(z^2 - a^2) + \dots + a_n(z^n - a^n)$$

$$= (z-a) Q(z)$$

where $Q(z)$ is a polynomial of degree $\leq n-1$.

So result follows by induction. $\boxed{\text{QED}}$

We often consider a new sequence of functions

$$S_1(z) = u_1(z)$$

$$S_2(z) = u_1(z) + u_2(z)$$

$$S_3(z) = u_1(z) + u_2(z) + u_3(z)$$

$$S_n(z) = \sum_{k=1}^{n} u_k(z)$$

If $\lim_{n\to\infty} S_n(z) = S(z)$ exists

we say that the infinite series

$$\sum_{k=1}^{\infty} u_k(z) \qquad \left(\begin{array}{c}\text{infinite}\\ \text{series}\end{array}\right)$$

is convergent.

If the series converges for all z in some region R, we call R the region of convergence of the series.

Chapter VI : Infinite series

Let $u_1(z)$, $u_2(z)$, $u_3(z)$, ...
be single-valued functions of z
in some region.

We write

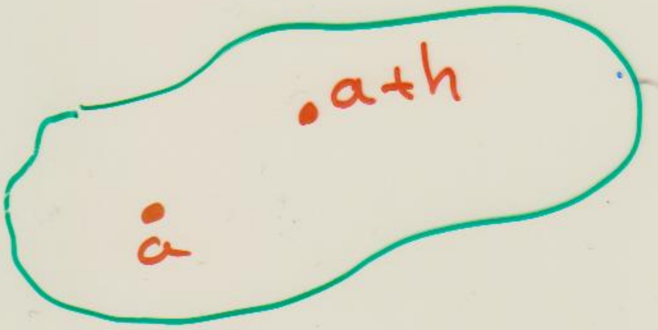
$$\lim_{n\to\infty} u_n(z) = U(z)$$

if $U(z)$ is some function such
that for any real $\varepsilon > 0$
there exist an integer N
(depending on ε and z) such
that

$$|U(z) - u_n(z)| < \varepsilon \quad \text{for } n > N.$$

Taylor's Theorem

Let $f(z)$ be analytic inside and on a simple closed curve C. Let a and $a+h$ be two points inside C.



Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots .$$

Alternative, writing $z = a+h$, $h = z-a$

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

The region of convergence is given by

$$\mathcal{R} = \{ z \in \mathbb{C} : |z-a| < R \}$$

where the radius of convergence R

is the distance from a to the nearest singularity of $f(z)$.

On the circle $|z-a| = R$ the series may or may not converge.

For $|z-a| > R$ the series diverges.

Example Let $f(z) = \ln(1+z)$ where we consider the branch with $f(0) = 0$.

Expand $f(z)$ in a Taylor's series about $z=0$ and determine the radius of convergence.

Sol^n

$f(z) = \ln(1+z)$ $\qquad$ $f(0) = 0$

$f'(z) = \frac{1}{1+z} = (1+z)^{-1}$ $\qquad$ $f'(0) = 1$

$f''(z) = \frac{-1}{(1+z)^2} = -(1+z)^{-2}$ $\qquad$ $f''(0) = -1$

$f'''(z) = 2(1+z)^{-3}$ $\qquad$ $f'''(0) = 2!$

$\vdots$

$f^{(n)}(z) =$

$f^{(n)}(0) = (-1)^{n-1}(n-1)!$

$f(z) = \ln(1+z)$

$= f(0) + f'(0)\,z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \dots$

$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots -$

Radius of convergence ?