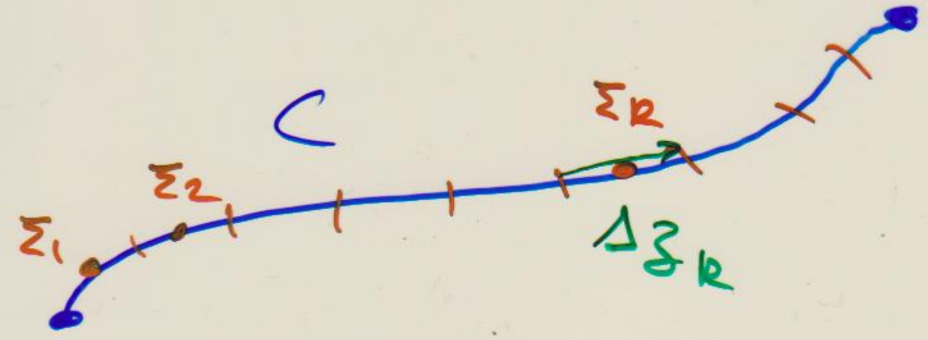
<u>Lemma</u> If $f(z)$ is integrable along a curve C of finite length L, and if there is a positive number M such that $|f(z)| \leq M$ on C, then

$$\left| \int_C f(z)\, dz \right| \leq ML .$$



Proof

$$\left| \int_C f(z)\, dz \right| = \left| \lim_{n\to\infty} \sum_{k=1}^{n} f(\varepsilon_k)\, \Delta z_k \right|$$

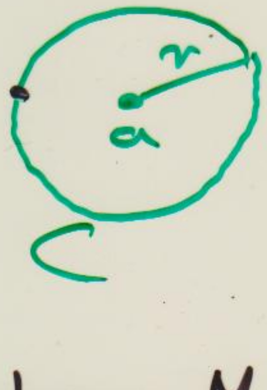
$$\left|\sum_{k=1}^{n} f(\xi_k)\,\Delta z_k\right| \le \sum_{k=1}^{n} |f(\xi_k)|\,|\Delta z_k|$$

$$\le M \sum_{k=1}^{n} |\Delta z_k|$$

$$\le ML.$$

Cauchy's Inequality

Suppose that $f(z)$ is analytic inside and on a circle C of radius r and centre $z = a$. Then



$$|f^{(n)}(a)| \le \frac{M \cdot n!}{r^n}, \quad n = 0, 1, 2, \ldots$$

where M is an upper bound of $|f(z)|$ on C.

Proof Cauchy's integral formulae are:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}$$

$$n = 0, 1, 2, 3, \ldots$$

On C we have $|z-a| = r$ and C has length $L = 2\pi r$

so

$$|f^{(n)}(a)| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z-a)^{n+1}} \right|$$

$$\leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r$$

$$= \frac{n! \cdot M}{r^n}$$

Liouville's Theorem

Suppose we have a function $f(z)$ and constant M such that, for all $z \in \mathbb{C}$,

i) $f(z)$ is analytic

ii) $|f(z)| \leq M$.

Then $f(z)$ must be a constant.

Proof Cauchy's inequality with $n=1$ is

$$|f'(z)| \leq \frac{M}{r}.$$

Letting $r \to \infty$ we see that $|f'(z)| = 0$. Hence $f'(z) = 0$.

Hence $f(z)$ is a constant.

$\boxed{\text{III}}$

Fundamental Theorem of Algebra

Any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

with $a_n \neq 0$, $n \geq 1$ has at least one root in $\mathbb{C}$.

Proof suppose $P(z)$ has no root in $\mathbb{C}$. Then

$$f(z) = \frac{1}{P(z)}$$

would be analytic for all $z \in \mathbb{C}$.

Also $|f(z)| = \dfrac{1}{|P(z)|}$ would be bounded. So by Liouville's Theorem $f(z) = \dfrac{1}{P(z)}$ would be constant, Contradiction.

$\boxed{\text{MJ}}$