Second Test: Monday 4th March

3rd & 4th Tests: Monday 25th March

Second Corollary to Cauchy's Theorem:

<u>Corollary</u> Let $f(z)$ be analytic in a simply connected region $R$. Then

$$F(z) := \int_a^z f(u)\,du$$

is analytic and $F'(z) = f(z)$.

<u>Example</u> Let $C$ be the curve $y = x^3 - 3x^2 + 4x - 1$ joining $(1,1)$ and $(2,3)$ in $\mathbb{R}^2 = \mathbb{C}$.

Evaluate

$$\int_C (12z^2 - 4iz)\,dz$$

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N.B. $f(z) = 12z^2 - 4iz$ is analytic.

Define

$$F(z) := \int_{a=1+i}^{z} (12z^2 - 4iz)\, dz$$

Need to calculate $F(2+3i)$.

Let $G(z)$ be some function such that $G'(z) = 12z^2 - 4iz$.

$$\boxed{\begin{aligned} a &= 1+i \\ b &= 2+3i \end{aligned}}$$

Then

$$G'(z) = F'(z)$$

and so

$$G(z) = F(z) + k.$$

Now

$$G(a) = 0 + k$$

$$G(b) = F(b) + k$$

So $F(b) = G(b) - G(a)$

Let's take

$$G(z) = 4z^3 - 2iz^2$$

Then

$$\int_{1+i}^{2+3i} (12z^2 - 2iz)\,dz = 4z^3 - 2iz^2 \Big|_{1+i}^{2+3i}$$

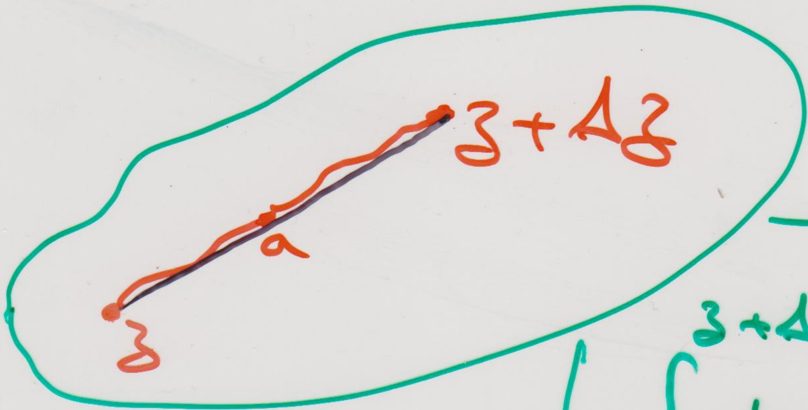
$$= -156 + 38i.$$

Proof of corollary

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)$$

$$= \frac{1}{\Delta z}\int_a^{z+\Delta z} f(u)\,du - \frac{1}{\Delta z}\int_a^{z} f(u)\,du - f(z)$$

$$\frac{1}{\Delta z}\int_z^{z+\Delta z} f(u)\,du \quad - \quad f(z)$$




Cauchy's Theorem

$$\int_z^{z+\Delta z} k\,du$$

$$= k\int_z^{z+\Delta z} du$$

$$= k\,\Delta z$$

$$\frac{1}{\Delta z}\int_z^{z+\Delta z} f(u)\,du \quad - \quad \frac{1}{\Delta z}\int_z^{z+\Delta z} f(z)\,du$$

$$\frac{1}{\Delta z}\int_z^{z+\Delta z} \Big( f(u) - f(z) \Big)\,du \qquad (*)$$

Since $f(z)$ is continuous,

$|f(u) - f(z)| < \varepsilon$ provided

$|u - z| < \Delta z$.

Also

$$\left| \int_z^{z+\Delta z} (f(u) - f(z))\, du \right|$$

$$< \varepsilon |\Delta z| . \qquad (**)$$

Combining $(*)$ with $(**)$:

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right|$$

$$= \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} (f(u) - f(z))\, du \right| < \varepsilon .$$

This means

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z)$$

# Chapter V

## Cauchy's Integral Formulae

Last lecture we showed that for any simple closed curve $C$ and any $a$ inside $C$


we have

$$\oint_C \frac{dz}{z-a} = 2\pi i$$

This example is routinely extended to

$$\oint_C \frac{dz}{(z-a)^n} = 0 \qquad \text{for } n = 2, 3, 4, \dots \quad n \neq 1$$

The method of calculation
xtends to:

<u>Theorem</u> Let $f(z)$ be analytic
side and on a simple
losed curve $C$, and let $a$
e any point inside $C$.

Then

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

and

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{1}{n!}2\pi i\, f^{(n)}(a)$$

for $n = 1, 2, 3, \ldots$

<u>Remark</u> If $f(z)$ is analytic inside and on $C$, then its value on $C$ completely determines its value

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)}\, dz$$

at any point $a$ in $C$.