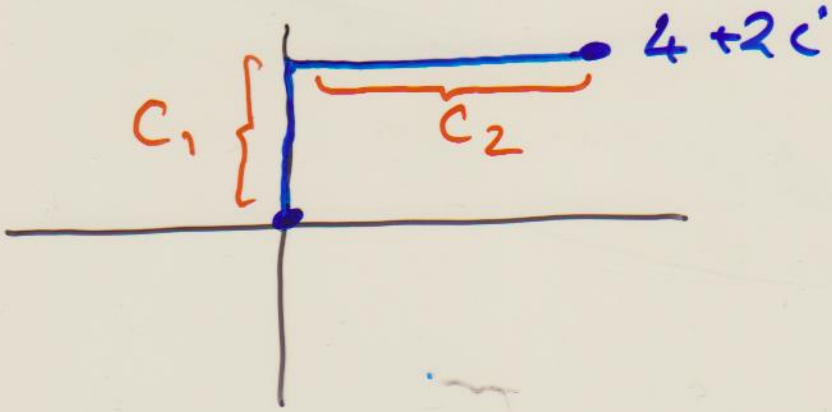
Example Evaluate $\int_c \bar{z}\, dz$

for C the curve



Sol^n

$$\int_c f(z)\,dz = \int_c (u+iv)(dx+idy)$$

$$= \int_c u\,dx - v\,dy + i\int_c v\,dx + u\,dy$$

$$\int_c \bar{z}\,dz = \int_c x\,dx + y\,dy + i\int_c -y\,dx + x\,dy$$

on C_1 : $x=0, dx=0, y=t, dy=dt$

$$\int_{C_1} \bar{z}\,dz = \int_0^2 t\,dt + i\int_0^2 0 = \frac{t^2}{2}\Big|_0^2 = 2$$

On C_2: $y=2$, $dy=0$, $x=t$, $dx=dt$

$$\int_{C_2} \bar{z}\,dz = \int_0^4 t\,dt + i\int_0^4 -2\,dt$$

$$= \left.\frac{t^2}{2}\right|_0^4 + i\left(-2t\Big|_0^4\right)$$

$$= 8 + -8i$$

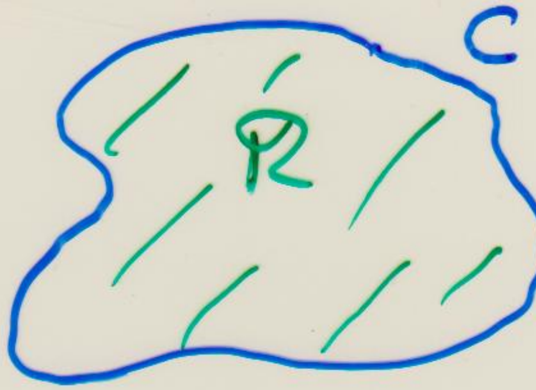
Thus

$$\int_C \bar{z}\,dz = \int_{C_1} \bar{z}\,dz + \int_{C} \bar{z}\,dz = \underline{\underline{10-8i}}$$

Cauchy's Theorem

Let $f(z)$ be analytic in a region R and on the boundary C of R.

Then



$$\int_C f(z)\,dz = 0.$$

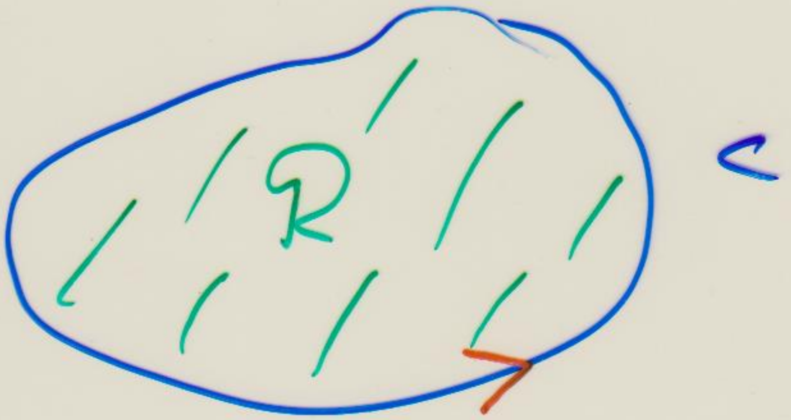
Proof See problem's 13, 14, 15, 16 for Goursat's proof.

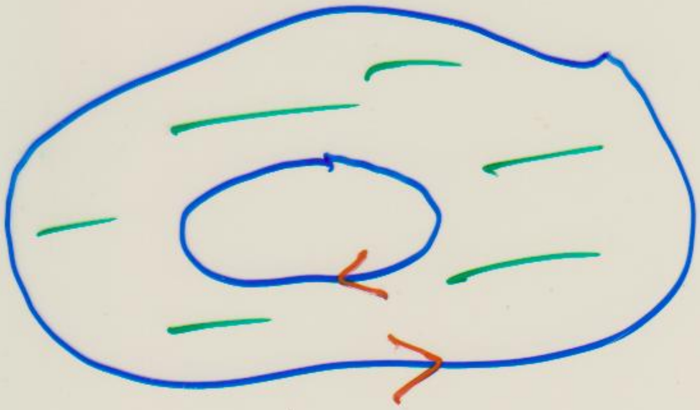
Notation Let C be a closed curve bounding a region R. To travel around C is the positive direction means that we keep the region R to our left.

we write

$$\oint_C f(z)\, dz$$

to mean the integral, travelling around C in the positive direction



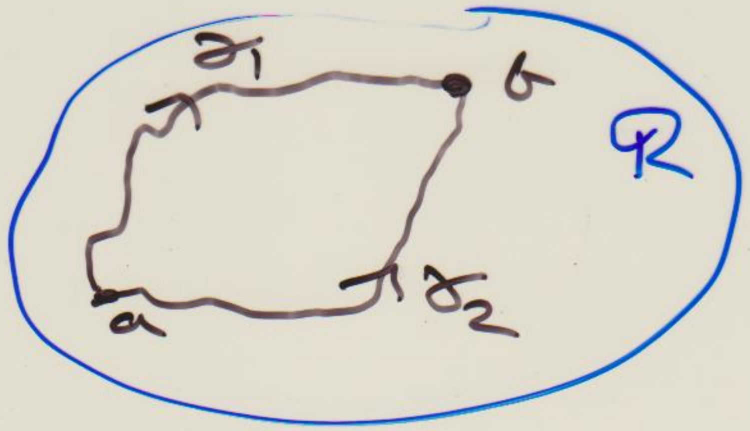


Corollary to Cauchy's Theorem

If $f(z)$ is analytic on a simply connected region R then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

for any two curves γ_1, γ_2 in R starting at $a \in R$ and ending at $b \in R$.



Proof

Cauchy's Theorem says

$$0 = \int_{aFbEa} f(z)\,dz$$

$$= \int_{\gamma_2} f(z)\,dz + \int_{bEa} f(z)\,dz$$

$$= \int_{\gamma_2} f(z)\,dz - \int_{\gamma_1} f(z)\,dz$$

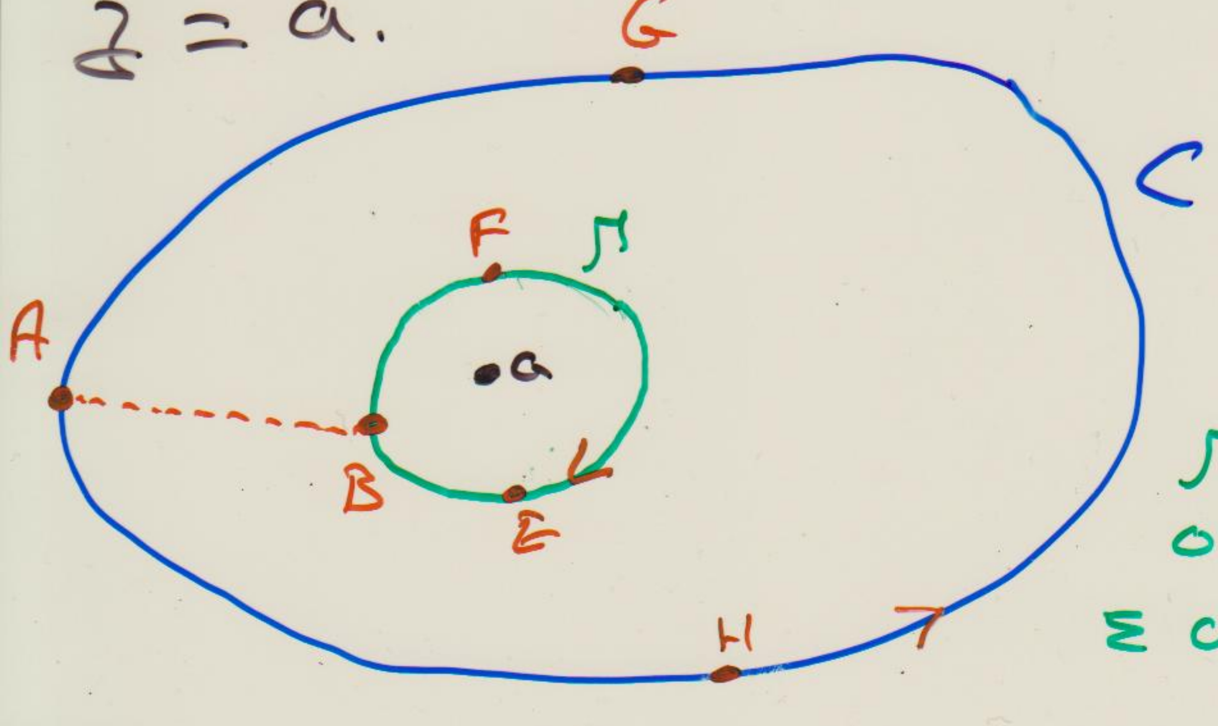
$\boxed{\!/\!/\!}$

Problem Evaluate

$$\oint_C \frac{dz}{z-a}$$

where C is any simple closed curve and where a is any point in the region bounded by C.

Sol^n $f(z) = \dfrac{1}{z-a}$ is analytic everywhere except at $z = a$.



Γ = circle of radius ε about a.

$$0 = \int_{AHGABFEBA} \frac{1}{z-a}\, dz$$

$$= \oint_{C} \frac{1}{z-a}\, dz + \int_{AB} \frac{1}{z-a}\, dz + \oint_{\gamma} \frac{1}{z-a}\, dz + \int_{BA} \frac{1}{z-a}\, dz$$

$$= \oint_{C} \frac{1}{z-a}\, dz + \oint_{\gamma} \frac{1}{z-a}\, dz$$

Hence

$$\oint_{C} \frac{1}{z-a}\, dz = \oint_{\gamma} \frac{1}{z-a}\, dz$$

On Γ, $|z-a| = \varepsilon$

or $z - a = \varepsilon e^{i\theta}$

or $z = a + \varepsilon e^{i\theta}$, $0 \le \theta < 2\pi$

So

$$\int_{\Gamma} \frac{1}{z-a}\, dz = i \int_0^{2\pi} \frac{\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}}\, d\theta$$

$$= i \int_0^{2\pi} d\theta = 2\pi i.$$

Hence

$$\boxed{\oint_C \frac{dz}{z-a} = 2\pi i}$$