Example Evaluate $\int_C \bar{z}\, dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by

$z = t^2 + it$.

Sol^n $t = 0$ at $z = 0$

$t = 2$ at $z = 4 + 2i$

$$\int_C \bar{z}\, dz = \int_0^2 \overline{(t^2 + it)}\, d(t^2 + it)$$

$$= \int_0^2 (t^2 - it)(2t + i)\, dt \quad \left(\begin{array}{c}\text{Trust} \\ \text{me} \\ \text{here}\end{array}\right)$$

$$= \int_0^2 2t^3 + t - it^2\, dt$$

$$= \frac{2t^4}{4}\Big|_0^2 + \frac{t^2}{2}\Big|_0^2 - i\frac{t^3}{3}\Big|_0^2$$

$= 8 + 2 - i\frac{8}{3}$

$= 10 - \frac{8i}{3}$.

Real Line Integrals

Given a smooth curve

$\gamma : [a, b] \longrightarrow \mathbb{R}^2$,

$t \longmapsto \gamma(t) = (x(t), y(t))$,

and given a continuous function

$F : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \longmapsto (P(x,y), Q(x,y))$

we define the line integral of F over γ to be

$\int_\gamma P\,dx + Q\,dy :=$

$$\int_\gamma P\,dx + Q\,dy := \int_a^b P(x(t), y(t))\, x'(t)\, dt + \int_a^b Q(x(t), y(t))\, y'(t)\, dt.$$

If we think of $F(x,y)$ as a force exerted on a particle, then $\int_\gamma P\,dx + Q\,dy$ is the work done by the force as the particle moves along the curve.

Connection between real and complex line integrals

For $f(z) = u(x,y) + i\,v(x,y)$

we have

$$\int_C f(z)\,dz = \qquad (*)$$

$$\int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

Example Evaluate $\int_C \bar{z}\,dz$

from $z=0$ to $z = 4+2i$

along the curve $z = t^2 + it$.

Sol^n $f(z) = x - iy$

$\qquad\qquad\quad P \qquad Q$

$$\int_C f(z)\,dz = \int_C x\,dx + y\,dy$$

$$+\, i\int_C -y\,dx + x\,dy$$

Now C is the curve

$(x(t), y(t)) = (t^2, t)$

from $t=0$ to $t=2$.

$$\int_C f(z)\,dz = \int_{t=0}^{2} t^2(2t\,dt) + t\,dt$$

$$+\, i\int_{t=0}^{2} (-t)(2t\,dt) + t^2\,dt$$

$$= \int_0^2 2t^3 + t\,dt + i\int_0^2 -t^2\,dt$$

$$= \frac{t^4}{2} + \frac{t^2}{2}\Big|_0^2 + i\left(-\frac{t^3}{3}\Big|_0^2\right) = 10 - \frac{8i}{3}$$

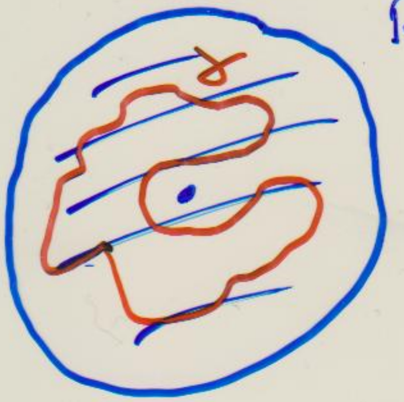
Simply and Multiply Connected regions

A region R is called <u>simply connected</u> if any simple closed curve $\gamma : [a, b] \to R$ can be "shrunk to a point" without leaving the region.

closed curve $\Leftrightarrow$ $\gamma(a) = \gamma(b)$

simple curve $\Leftrightarrow$ γ does not intersect itself (except at $\gamma(a) = \gamma(b)$)

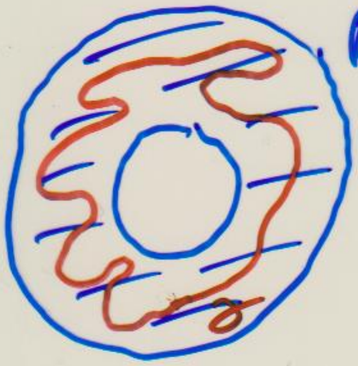
R_1



$R_1 = \{ z \in \mathbb{C} : |z| \le 1 \}$

Simply connected

R_2



$R_2 = \{ z \in \mathbb{C} : \frac{1}{2} \le |z| \le 1 \}$

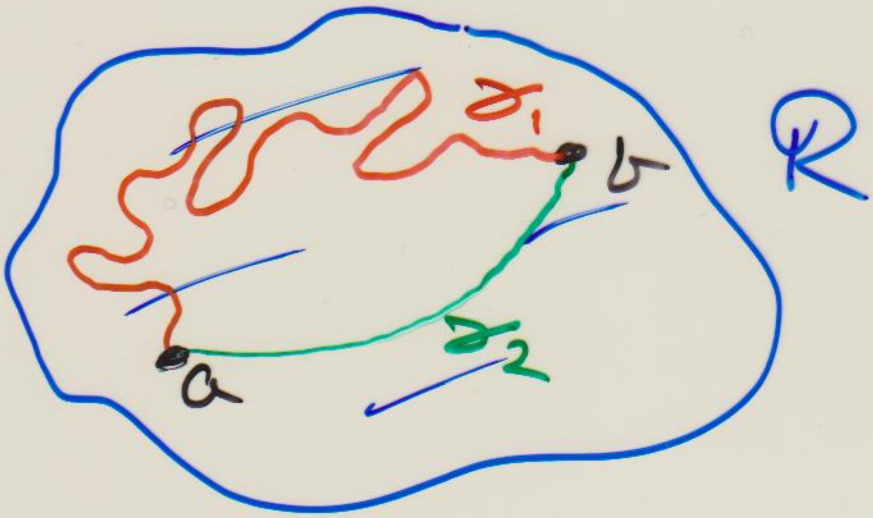
Not simply connected

not simply connected

Theorem (A consequence of Cauchy's Theorem)

Let $f(z)$ be analytic on a simply connected region R. Let $a, b \in R$.

Let γ_1 and γ_2 be two paths in R from a to b.



Then

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz$$