Complex Variables (MA287)

Text: "Complex Variables"
by Seymour Lipschutz
(Schaum's outline series)

€ 14.45

Course outline

chapters 1-8

2/3 lectures per chapter

four 50-minute tests worth 30%

Chapter One: Complex numbers

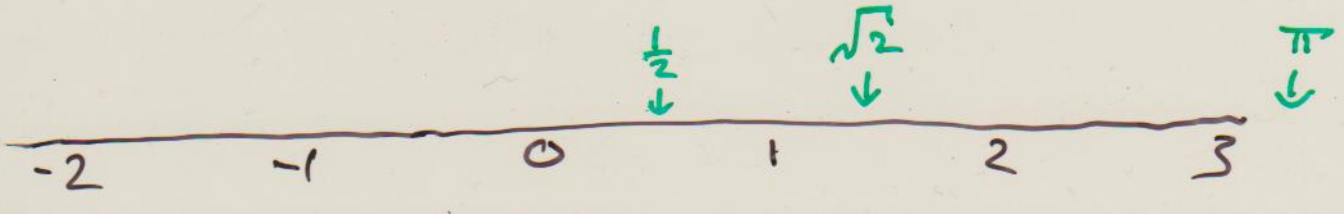
$\mathbb{N} = \{1, 2, 3, \cdots\}$

$\mathbb{N} \cup \{0\}$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$

$\mathbb{Q} = \{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \}$

$\mathbb{R}$ = rational numbers $\cup$ irrational numbers



$\mathbb{C}$ = complex numbers

$= \{ x + yi \;:\; x, y \in \mathbb{R} \}$

where $i^2 = -1$.

Addition:

$$(2+3i) + (1+2i) = 3 + 5i$$

subtraction:

$$(2+3i) - (1+2i) = 1 + i$$

multiplication:

$$(2+3i)(1+2i) =$$

$$2.1 + 2.2i + 1.3i + 3.2.i^2$$

$$= -4 + 7i$$

Division:

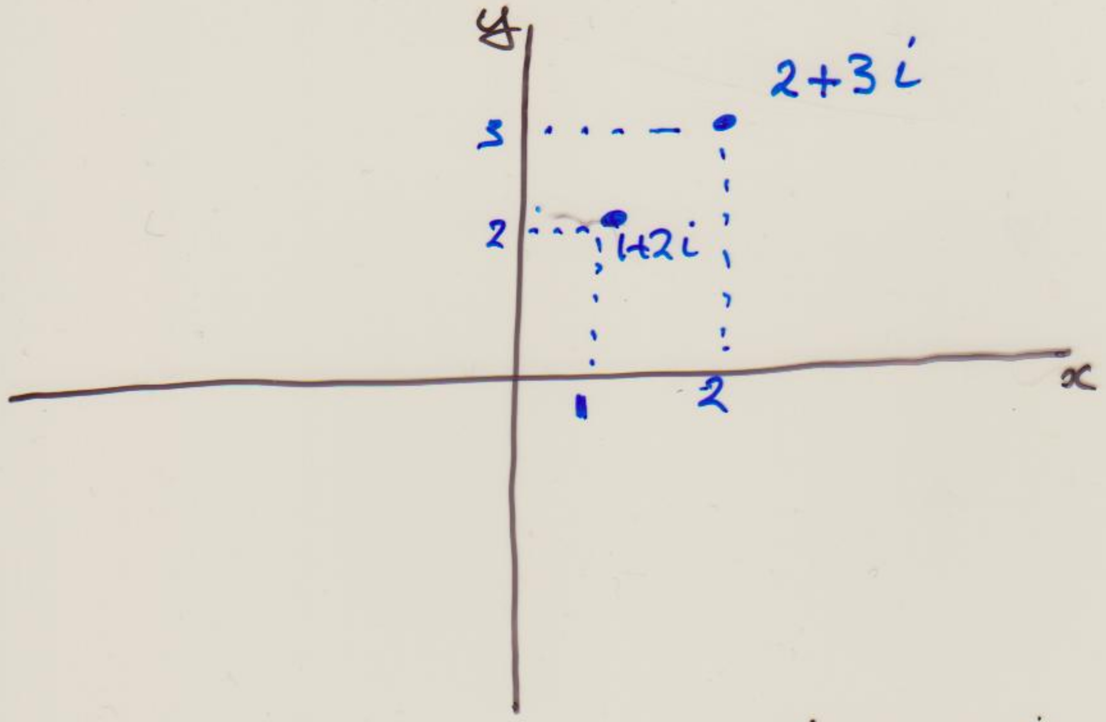
$$\frac{2+3i}{1+2i} = \frac{(2+3i)}{(1+2i)}\frac{(1-2i)}{(1-2i)}$$

$$= \frac{2 - 6i^2 - 4i + 3i}{1 - (2i)^2}$$

$$= \frac{8 - i}{5}$$

$$= \frac{8}{5} - \frac{1}{5} i$$

Graphical Representation



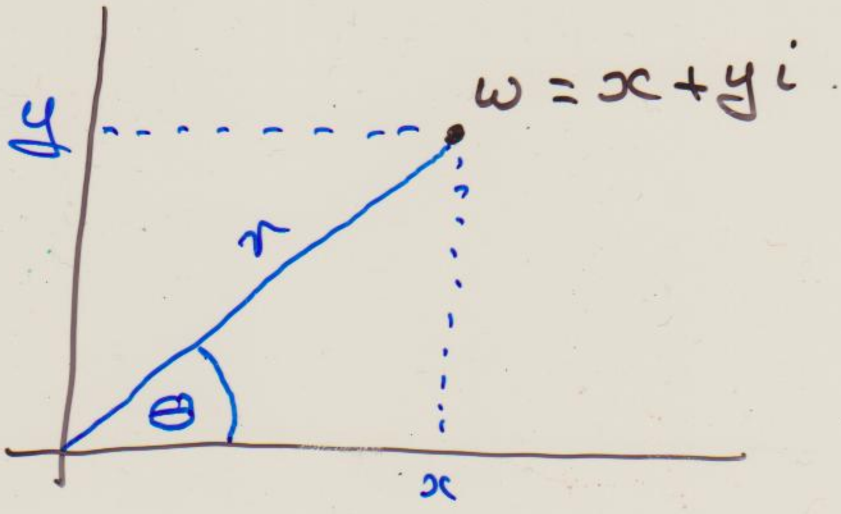
Terminology : x is real axis

y is imaginary axis

Polar form of complex Numbers

$x = r\cos\theta$

$y = r\sin\theta$



$$\boxed{w = x + iy}$$ cartesian form

$$\boxed{w = r(\cos\theta + i\sin\theta)}$$ polar form

<u>Defn</u> $|w| = r$ is the <u>modulus</u> of w.

$\arg(w) = \theta$ is the <u>argument</u> of w.

Convention: $0 \leq \theta < 2\pi$

<u>Theorem</u>

$$|w_1 w_2| = |w_1|\,|w_2|$$

$$\arg(w_1 w_2) = \arg(w_1) + \arg(w_2)$$

<u>Proof</u>

Let $w_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$w_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

Then

$$\omega_1 \omega_2 = r_1 r_2 (\cos\theta_1 + i \sin\theta_1) \cdot (\cos\theta_2 + i \sin\theta_2)$$

$$= r_1 r_2 \left(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i \{ \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 \} \right)$$

$$= r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right).$$

An immediate consequence is:

<u>De Moivre's Theorem</u>

If $z = r(\cos\theta + i \sin\theta)$

then

$$z^n = r^n (\cos(n\theta) + i \sin(n\theta)).$$

Roots of Unity

If $z^n = 1$ then we say that z is an n^{th} root of unity.

Problem List all 4^{th} roots of unity.

Soln Suppose $z^4 = 1$ and that $z = r(\cos\theta + i\sin\theta)$.

Then

$$z^4 = r^4(\cos 4\theta + i \sin 4\theta) = 1.$$

So $r^4 = 1$. Hence $r = 1$.

Also

$$\cos 4\theta = 1$$

$$\sin 4\theta = 0.$$

The only solutions are:

$$z = \cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4}$$

with $k = 0, 1, 2, 3$.

The only solutions are

$$z_0 = \cos 0 + i\sin 0 = 1$$

$$z_1 = \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4} = i$$

$$z_2 = \cos\pi + i\sin\pi = -1$$

$$z_4 = \cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4} = -i.$$

Observe:

$$z_0 + z_1 + z_2 + z_3$$

$$= 1 + i + (-1) + (-i) = 0.$$