Determinants

Today A, B denote $n \times n$ matrices
(or 3×3 matrices)

It is possible to define a
number

$$|A| = \det(A)$$

such that the results given
last time (for 2×2 matrices) hold.
In particular:

Result 3

The row operation

$$A \xrightarrow[R_i \mapsto R_i + aR_j]{} B \qquad (i \neq j)$$

does not change the determinant.

Result 4

The row operation

$$A \xrightarrow[R_i \to aR_i]{} B$$

has the effect $|B| = a|A|$

Result 5

The row operation

$$A \xrightarrow[R_i \leftrightarrow R_j]{} B \qquad (i \neq j)$$

changes the determinant by a factor of -1.

Result 6

If all entries in A $\left\{\begin{matrix}\text{above}\\ \text{below}\end{matrix}\right.$ the diagonal are zero then $|A|$ = product of the diagonal entries.

Illustration

$$\left|\begin{pmatrix} 2 & 7 & 107 \\ 0 & 3 & 119 \\ 0 & 0 & 4 \end{pmatrix}\right| = 2 * 3 * 4$$

Example Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & -4 \end{pmatrix}.$$

Find $|A|$.

Sol^n

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & -4 \end{vmatrix} \xrightarrow[R_3 \mapsto R_3 - 2R_1]{R_2 \mapsto R_2 - R_1}$$

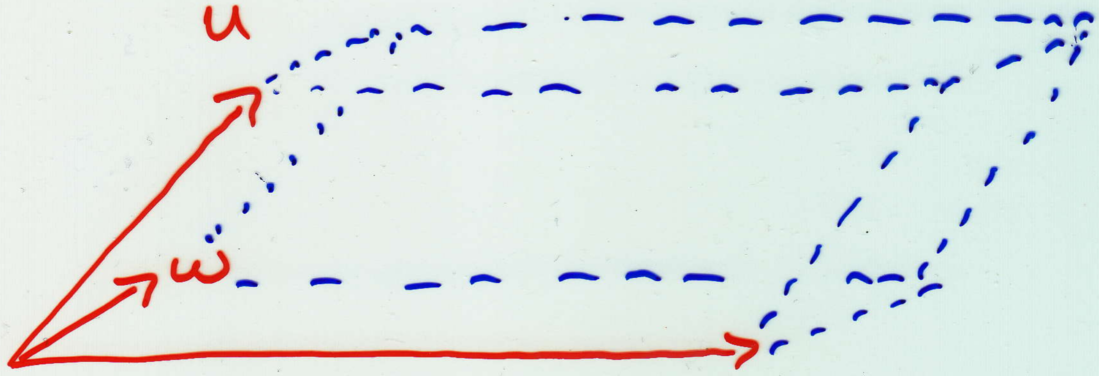
$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -6 \end{vmatrix} \xrightarrow{R_3 \mapsto R_3 - R_1}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -7 \end{vmatrix} = 1 \times 1 \times (-7).$$

$$= -7$$

Example what is the volume of the parallel^{pd} defined by the vectors

$$u = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$



$$\text{Area} = \pm \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & -4 \end{vmatrix} = \pm -7$$

$$\text{Area} = 7$$

Example Let $a, b, c > 0$

Consider

$$A = \begin{pmatrix} a & b & a^{-1} \\ a^2 & b & a^{-2} \\ a^3 & b & a^{-3} \end{pmatrix}$$

Find $|A|$.

Sol^n

$$|A| \xlongequal[R_3 \mapsto R_3 - a^2 R_1]{R_2 \mapsto R_2 - a R_1} \begin{vmatrix} a & b & a^{-1} \\ 0 & (1-a)b & a^{-2}-1 \\ 0 & (1-a^2)b & a^{-3}-a \end{vmatrix}$$

$$\xlongequal{R_3 \mapsto R_3 - (1+a)R_2} \begin{vmatrix} a & b & a^{-1} \\ 0 & (1-a)b & a^{-2}-1 \\ 0 & 0 & a^{-3}-a-(1+a)(a^{-2}-1) \end{vmatrix}$$

$$= a(1-a)b\,(a^{-3} - a^{-2} - a^{-1} + 1)$$

$$= b(1-a)(a^{-2} - a^{-1} - 1 + a)$$

$\stackrel{?}{=} \frac{b}{a^2}(1+a)(1-a)^3$

Formula for the inverse of a matrix

For any $n \times n$ matrix A one can define an adjoint matrix $\mathrm{adj}(A)$ such that

$$\boxed{A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A).}$$

We won't define the adjoint. However, the formula implies that A has no inverse precisely when $|A| = 0$.

Exercise is the following matrix invertible?

$$A = \begin{pmatrix} 4 & 0 & 2 & -3 \\ 2 & 1 & 0 & -1 \\ 0 & 0 & 0 & -5 \\ 2 & 0 & -4 & -4 \end{pmatrix}.$$

Sol^n

$$|A| \underset{R_3 \Leftrightarrow R_4}{=} - \begin{vmatrix} 4 & 0 & 2 & -3 \\ 2 & 1 & 0 & -1 \\ 2 & 0 & -4 & -4 \\ 0 & 0 & 0 & -5 \end{vmatrix}$$

$$\underset{R_1 \Leftrightarrow R_3}{=} \begin{vmatrix} 2 & 0 & -4 & -4 \\ 2 & 1 & 0 & -1 \\ 4 & 0 & 2 & -3 \\ 0 & 0 & 0 & -5 \end{vmatrix}$$

$$\underset{\substack{R_2 \mapsto R_2 - R_1 \\ R_3 \mapsto R_3 - 2R_1}}{=} \begin{vmatrix} 2 & 0 & -4 & -4 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 10 & 5 \\ 0 & 0 & 0 & -5 \end{vmatrix}$$

$= 2 \times 1 \times 10 \times (-5)$

$= -100$

Hence $|A| \neq 0$.

Hence A is invertible.