Matrix multiplication of a 3×3 matrix

$$A = \begin{pmatrix} \bullet\!\!-\!\!-\!\!\overset{R_1}{\bullet}\!\!-\!\!-\!\!\bullet \\ \bullet\!\!-\!\!-\!\!\overset{R_2}{\bullet}\!\!-\!\!-\!\!\bullet \\ \bullet\!\!-\!\!-\!\!\overset{R_3}{\bullet}\!\!-\!\!-\!\!\bullet \end{pmatrix}$$

and a 3×1 matrix

$$C = \begin{pmatrix} \bullet \\ \vert \\ \bullet \\ \vert \\ \bullet \end{pmatrix}$$

We define

$$\underset{(3\times\cancel{3})}{A}\ \underset{(\cancel{3}\times 1)}{C} = \underset{(3\times 1)}{\begin{pmatrix} R_1 C \\ R_2 C \\ R_3 C \end{pmatrix}}$$

The system

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}}_{B} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 21 \\ 31 \end{pmatrix}$$

implies

$$AB\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} 10 \\ 21 \\ 31 \end{pmatrix}$$

where

$$A = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

so $$I\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} 10 \\ 21 \\ 31 \end{pmatrix}$$

and

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 10 \\ 21 \\ 31 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

So $x = 3$

$y = 2$

$z = 1$

Question: How did we find the appropriate matrix A so that the above method worked?

Terminology

The inverse of 2 is $\frac{1}{2}$.

Fancy language: $2^{-1} = \frac{1}{2}$.

This is because $\frac{1}{2} \times 2 = 1$

Analogous Terminology

The inverse of a 3×3 matrix A is a matrix A^{-1} such that

$$A A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A^{-1} A$$

Question Repeated: How do we find the inverse of a matrix?

Gauss-Jordan method for finding the inverse of a matrix B

$$\left(B \;\vdots\; I \right) \xrightarrow[\text{operations}]{\text{row}} \left(I \;\vdots\; A \right)$$

Then $A = B^{-1}$.

Example

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 8 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 2 & -3 & -3 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_2]{}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right) \xrightarrow[R_3 \to -R_3]{}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right) \xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - 3R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -6 & 3 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right) \xrightarrow[R_1 \to R_1 - 2R_2]{}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -12 & 5 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right)$$

So $B^{-1} = \begin{pmatrix} 10 & -12 & 5 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}$