A function $f: X \to Y$ is often represented by its graph

$$\{(x, f(x)) : x \in X\} \subseteq X \times Y$$

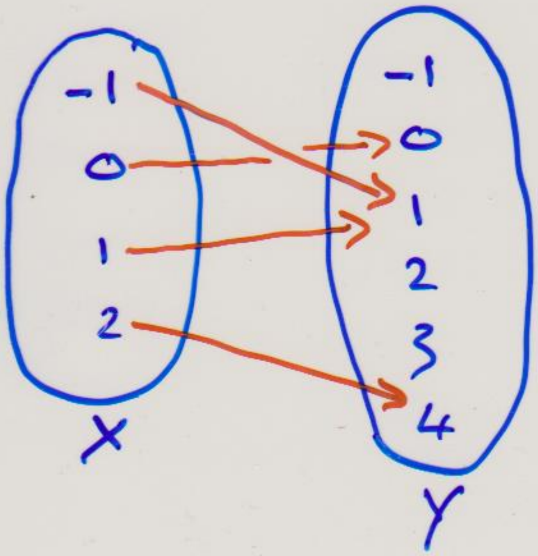
<u>Example</u> For $X = \{-1, 0, 1, 2\}$
$Y = \{-1, 0, 1, 2, 3, 4\}$

the set

$$\{(-1,1), (0,0), (1,1), (2,4)\} \subseteq X \times Y$$

represents the function

$$f : X \to Y, \quad x \mapsto x^2 .$$



Not surjective
Not injective

Relations

<u>Definition</u> A relation consists of a domain X, a codomain Y and a subset

$$R \subseteq X \times Y .$$

Example $X = \mathbb{N}$

$Y = \mathbb{Z}$

$R = \{(x,y) \in \mathbb{N} \times \mathbb{Z} \ : \ x = y^2\}$

so

$R = \{(0,0), (1,1), (1,-1), (4,2), (4,-2), \dots\}$

Note: R is not the graph of any function in this example, since ~~any fn~~ we'd need $f: X \to Y$ with $f(4) = 2$ and $f(4) = -2$.

Note A function can be defined as a relation $R \subseteq X \times Y$ with the property that for each $x \in X$ there is exactly one $y \in X$ for which $(x,y) \in R$.

Notation Given a relation $R \subseteq X \times Y$ we write

$$x R y$$

to mean

$$(x,y) \in R.$$

Example X = set of males in Ireland

Y = set of females in Ireland

xRy if y is the sister of x.

This is not a function.

We can write R also as

$$R = \{(x,y) : y \text{ is the sister of } x\}.$$

Functions are an important class of relation,

"Equivalence relations" are another important class.

Definition A relation

$$R \subseteq X \times X$$

is said to be an equivalence relation if the following three properties hold:

1) $x R x$ for all $x \in X$. (Reflexive)

2) if $x R y$ then $y R x$. (symmetric)

3) If $x R y$ and $y R z$ then $x R z$. (Transitive)

<u>Example</u> X = all people in Ireland
Y = all people in Ireland

$x R y$ means "x and y have the same father".

Let's check that this is an equivalence relation.

$x R x$: x has same father as x. ✓

$x R y$ implies $y R x$: if x has the same father as y then y has the same father as x. ✓

$x R y$ and $y R z$ implies $x R z$: If x has same father as y, and if y has the same father as z, then x has the same father as z. ✓

<u>Example</u> $X = \mathbb{Z}$

$Y = \mathbb{Z}$

Let xRy mean that $x-y$ is an integer multiple of 5.

Is this an equivalence relation?

1) xRx for all x (since $x-x = 0.5$)

2) if xRy then yRx (since, if $x-y = 5k$, then $y-x = 5(-k)$)

3) if xRy and if yRz then xRz

(since $x-y = 5k$, $y-z = 5m$ implies

$$x-z = (x-y)+(y-z) = 5k+5m = 5(k+m).)$$

If we have an equivalence relation on a set X, then for $x \in X$ we write

$$[x]$$

for the set of all elements of X related to x.

Example $X = \mathbb{Z}$

xRy means "$x-y$ is a multiple of 5"

$[0] = \{ \cdots -10, -5, 0, 5, 10, 15 \cdots \}$

$[1] = \{ \cdots, -9, -4, 1, 6, 11, \cdots \}$

$[2] = \{ \cdots, -8, -3, 2, 7, 12, \cdots \}$

$[3] = \{ \cdots, -7, -2, 3, 8, 13, \cdots \}$

$[4] = \{ \cdots, -6, -1, 4, 9, 14, \cdots \}$

$[5] = [0]$

$[6] = [1]$.