Let $A = \{1, 2, 3, \{1\}, \{1,2,3\}\}$

A has five elements.

Are the following statements true or false?

$1 \subseteq A$ FALSE (1 is not a subset)

$1 \in A$ TRUE (1 is an element of A)

$\{1\} \in A$ TRUE

$\{1\} \subseteq A$ TRUE

$\{3\} \in A$ FALSE

$\{3\} \subseteq A$ TRUE

Functions

A function $f: X \to Y$ consists of:

- a set X (called the domain of X)
- a set Y (called the codomain of Y)
- a rule that assigns to each $x \in X$ a unique element $f(x) \in Y$.

Example $\lfloor \ \rfloor : \mathbb{Q} \longrightarrow \mathbb{Z}$ is a function, called the floor function, where

$\lfloor x \rfloor$ = greatest integer $\leq x$.

e.g. $\left\lfloor \frac{22}{7} \right\rfloor = 3$

$\left\lfloor -\frac{22}{7} \right\rfloor = -4$

Example Let $X = \{0, 1, 2\}$

$Y = \{0, 1, 2, 3, 4\}$

$x \longmapsto f(x) = x^2$

This is a function

$f : X \to Y, \ x \longmapsto x^2$

Functions are often represented by their "graphs."

Definition For sets X, Y the direct product is the set

$$X \times Y = \{(x, y) : x \in X, \ y \in Y\}$$

Example $X = \{0,1,2\}$, $Y = \{0,1,2,3,4\}$

$$X \times Y = \{(0,0), (0,1), (0,2), (0,3), (0,4),$$
$$(1,0), (1,1), (1,2), (1,3), (1,4),$$
$$(2,0), (2,1), (2,2), (2,3), (2,4)\}$$

Note:

$$\text{Size}(X \times Y) = \text{Size}(X) \cdot \text{Size}(Y)$$

Def: Let $f: X \to Y$ be a function. Its graph is the subset of $X \times Y$ defined as:

$$\{(x, f(x)) : x \in X\}.$$

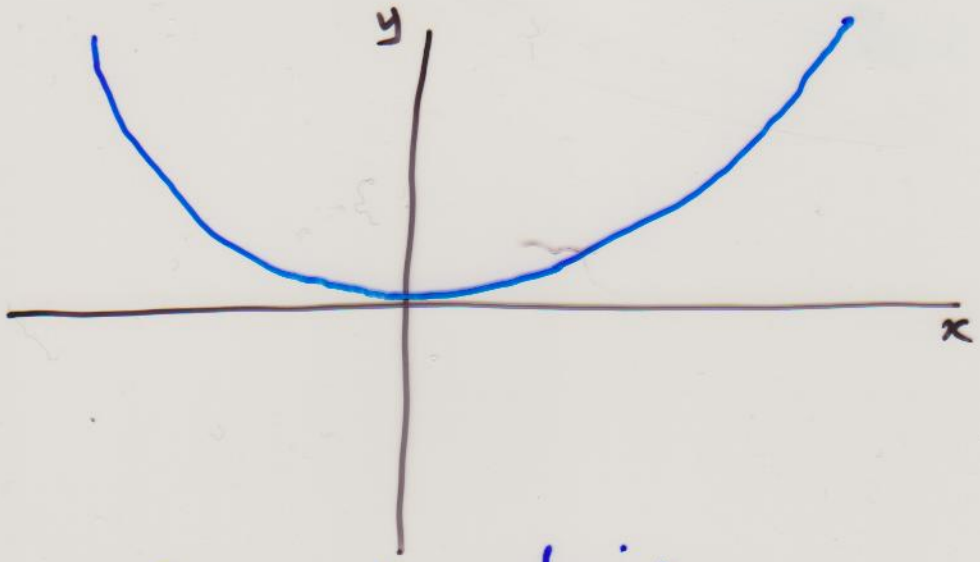
Example $f: \{0,1,2\} \to \{0,1,2,3,4\}, x \mapsto x^2$ has graph

$$\{(0,0), (1,1), (2,4)\}.$$

The function

$f : \mathbb{R} \to \mathbb{R}, \quad x \longmapsto x^2$

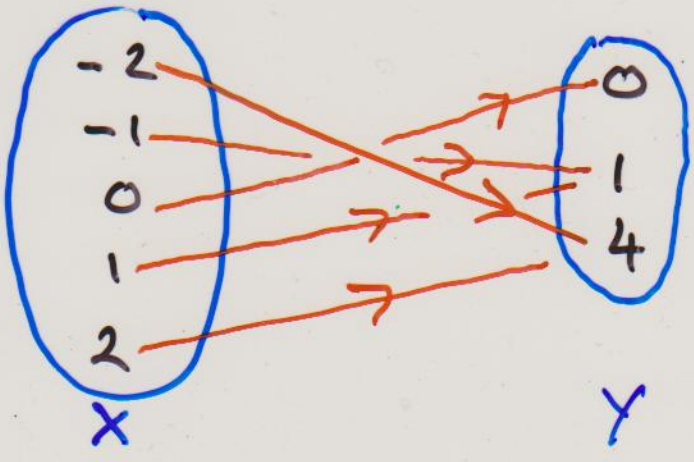
has graph a subset of $\mathbb{R} \times \mathbb{R}$ which we picture as



Surjective functions

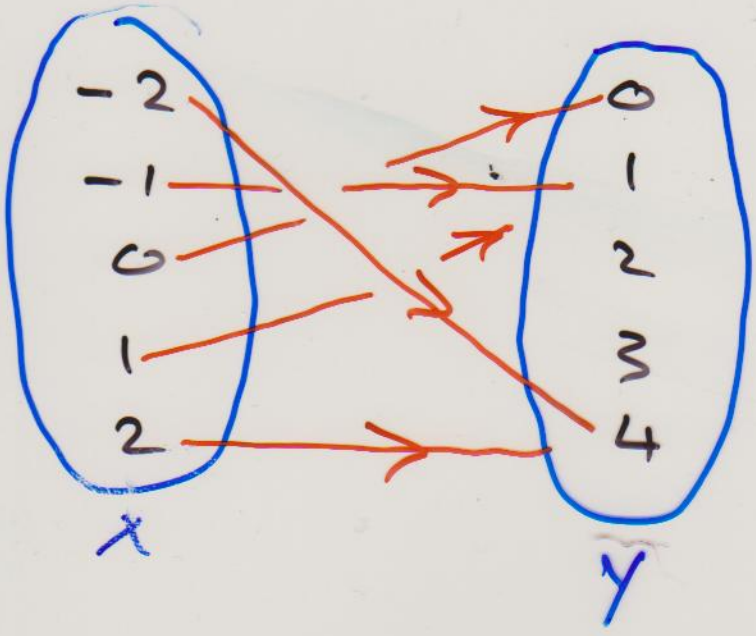
A function $f : X \to Y$ is <u>surjective</u> if, for each $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

<u>Example</u> $f : \{-2, -1, 0, 1, 2\} \to \{0, 1, 4\}, \quad x \mapsto x^2$



<u>Surjective</u>

Example $f: \{-2,-1,0,1,2\} \to \{0,1,2,3,4\}$,
$x \longmapsto x^2$



not surjective

Exercise:

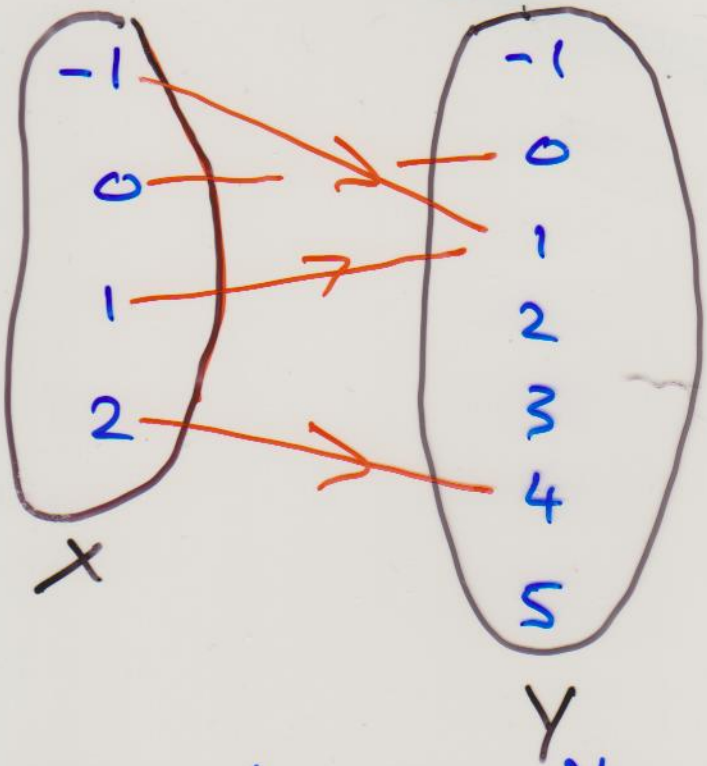
- is $f: \mathbb{R} \to \mathbb{R}$, $x \longmapsto x^3$ surjective ?
- is $f: \mathbb{Q} \to \mathbb{Q}$, $x \longmapsto x^3$ surjective ?

Injective Functions

A function $f: X \to Y$ is injective if $f(x) \neq f(x')$ whenever $x \neq x'$.

Example Consider

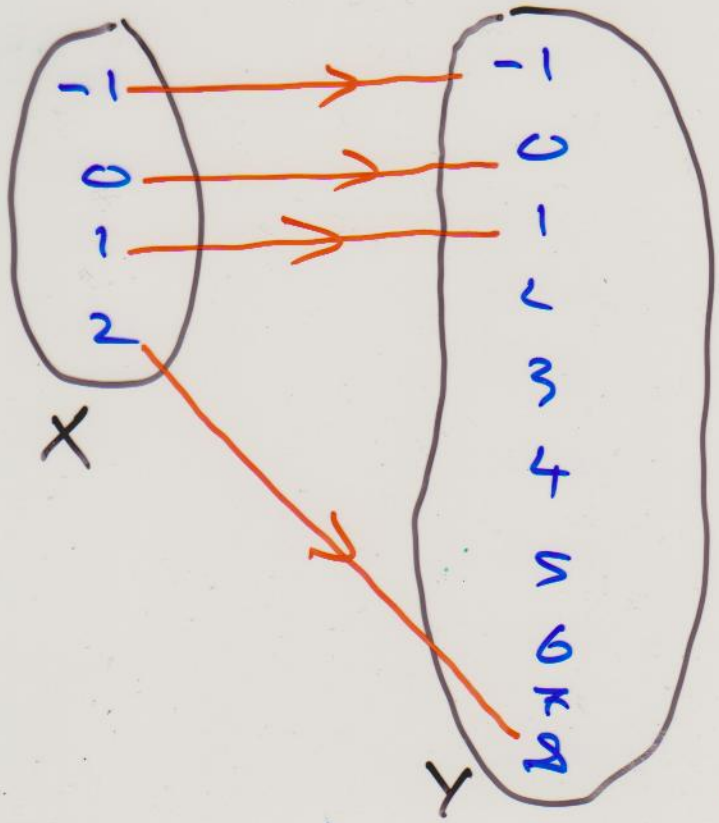
$f : \{-1, 0, 1, 2\} \longrightarrow \{-1, 0, 1, 2, 3, 4, 5\}, x \mapsto x^2$



not injective

Example Consider

$f : \{-1, 0, 1, 2\} \longrightarrow \{-1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}, x \mapsto x^3$



injective

Definition

A function $f: X \to Y$ is bijective if it is both injective and surjective.