Problem

Find the characteristic polynomial, the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} -1 & 4 & 0 \\ 0 & 3 & 0 \\ 8 & -12 & 7 \end{pmatrix}.$$

Hence (or otherwise) find a diagonal matrix D and a matrix E such that

$$E^{-1} A E = D.$$

Sol^n

Characteristic polynomial =

$|A - xI| =$

$$= \begin{vmatrix} -1-x & 4 & 0 \\ 0 & 3-x & 0 \\ 8 & -12 & 7-x \end{vmatrix}$$

$$\underset{R_1 \to R_1 - R_2}{=} \begin{vmatrix} -1-x & 1+x & 0 \\ 0 & 3-x & 0 \\ 8 & -12 & 7-x \end{vmatrix}$$

$$\underset{C_2 \to C_2 + C_1}{=} \begin{vmatrix} -1-x & 0 & 0 \\ 0 & 3-x & 0 \\ 8 & -4 & 7-x \end{vmatrix}$$

$$= -(1+x)(3-x)(7-x)$$

Characteristic polynomial of A

The eigenvalues of A are the solutions to the equation

$$|A - xI| = 0.$$

i.e. $-(1+x)(3-x)(7-x) = 0.$

The eigenvalues of A are

$$\lambda_1 = -1, \quad \lambda_2 = 3, \quad \lambda_3 = 7$$

Let's now find corresponding eigenvectors.

$\underline{\lambda_1 = -1}$

Need non-zero vector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

such that

$$Av = -1 . v$$

or $$Av + 1 . v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or $$\boxed{\left(A - (-1) I\right) v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

$$\begin{pmatrix} 0 & 4 & 0 \\ 0 & 4 & 0 \\ 8 & -12 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

One solution / eigenvector is

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\underline{\lambda_2 = 3}$

$$(A - \lambda_2 I)\, v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 4 & 0 \\ 0 & 0 & 0 \\ 8 & -12 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

One eigenvector is

$$v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\lambda_3 = 7$

$$(A - \lambda_3 I)\, v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -8 & 4 & 0 \\ 0 & -4 & 0 \\ 8 & -12 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

One eigenvector is

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Finally, set

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

Check

$$|E| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix}$$

$$\underset{R_1 \to R_1 - R_2}{=} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix} = 1 \neq 0.$$

Then E^{-1} exists and

$$E^{-1} A E = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Definition of a determinant for 2x2 and 3x3 matrices

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = +ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} =$$

$$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Example

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & -4 \end{vmatrix} = 1.\begin{vmatrix} 2 & 2 \\ 3 & -4 \end{vmatrix} - 1.\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} + 1.\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= -14 + 8 - 1 = -7.$$