Let A denote an $n \times n$ matrix.

Recall that a column vector v is an <u>eigenvector</u> of A with <u>eigenvalue</u> $\lambda \in \mathbb{R}$ if

1) $Av = \lambda v$
2) v is not the zero vector.

Eigenvectors are needed for many applications:

- Google page rank
- eigenfaces used by police

Recall from Semester I the basic method for finding eigenvectors/values.

Example Find the eigenvalues of

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Soln

Consider

$$\det(A - \lambda I) = 0$$

$$\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(2-\lambda) - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

So the eigenvalues of A are the solutions to this equation, namely

$$\lambda_1 = 1, \quad \lambda_2 = 3$$

We could use these eigenvalues to find eigenvectors.

To generalize this example to $n \times n$ matrices we first need

Determinants

Recall that for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we define the number

$$\det(A) = ad - bc.$$

If A is an $n \times n$ matrix (e.g. 3×3 matrix) it is possible to define a number

$$\det(A) = |A|$$

such that the following <u>four</u> properties hold.

Property 1

The row operation

$$A \xrightarrow[R_i \to R_i + \lambda R_j]{} B$$

does not change the determinant. $|A| = |B|$

Illustration

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow[R_1 \to R_1 + 2R_2]{} \begin{pmatrix} 7 & 10 \\ 3 & 4 \end{pmatrix} = B$$

$$|A| = 1.4 - 2.3 = -2$$

$$|B| = 7.4 - 3.10 = -2$$

Property 2

The row operation

$$A \xrightarrow[R_i \to aR_i]{} B$$

has the effect of increasing the determinant by a factor of a. $|B| = a|A|$

Illustration

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow[R_2 \to 3R_2]{} \begin{pmatrix} 1 & 2 \\ 9 & 12 \end{pmatrix} = B$$

$$|A| = -2$$

$$|B| = 1.12 - 2.9 = -6$$

Property 3

The row operation

$$A \xrightarrow[\substack{R_i \leftrightarrow R_j \\ i \neq j}]{} B$$

changes the sign of the determinant. $|B| = -|A|$

Illustration

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = B$$

$$|A| = -2$$

$$|B| = 3.2 - 1.4 = 2$$

Property 4

If all entries of A below the diagonal are zero, then

$\det(A)$ = product of diagonal entries of A

Illustration

$$A = \begin{pmatrix} 1 & 2012 \\ 0 & 3 \end{pmatrix}$$

$$|A| = 1.3 = 3$$

FACT One can define the number $\det(A)$ for $n \times n$ matrices in such a way that properties 1-4 hold.

Problem

Let
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & -4 \end{pmatrix}.$$

Calculate $\det(A)$.

Sol^n

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & -4 \end{vmatrix} \xrightarrow[\substack{R_2 \mapsto R_2 - R_1 \\ R_3 \mapsto R_3 - 2R_1}]{}$$

by Prop 1
$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -6 \end{vmatrix} \xrightarrow[R_3 \mapsto R_3 - R_2]{}$$

by Prop 1
$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -7 \end{vmatrix}$$

by Prop 4
$$= (1)(1)(-7) = -7$$