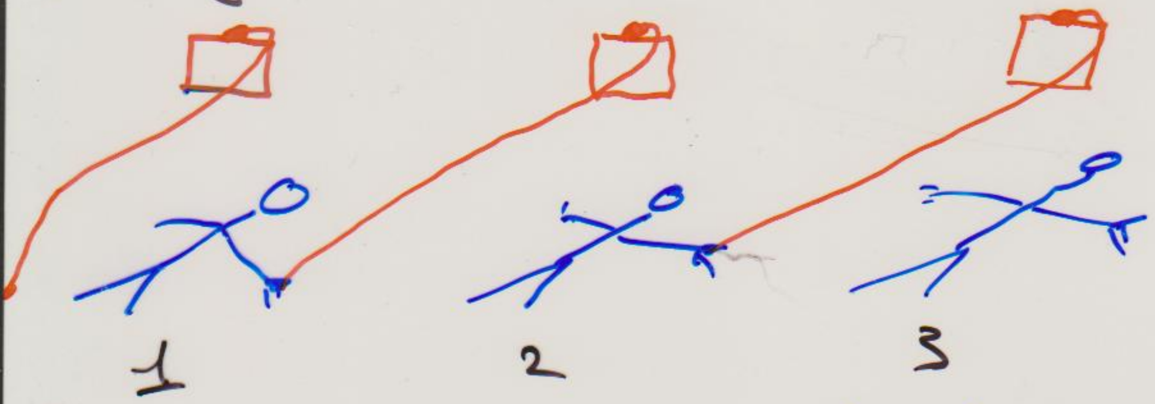
Principle of Induction

Version 1

Suppose n men have their heads in guillotines



with the nth man holding the rope for the guillotine for $n+1$st man.

Principle of Induction V1

If the first man loses his head then all men lose their head

Version 2

Let $P(n)$ be a proposition depending on an integer n.

Principle of induction v2.

if $P(1)$ is true,

and if

$\forall n, P(n) \longrightarrow P(n+1)$ (i.e. $P(n)$ implies $P(n+1)$)

is true,

then $P(n)$ is true for all n.

<u>Example</u> Let's prove that the proposition

$$P(n): \quad 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$P(1): \quad 1 = \frac{1(1+1)}{2}$$ clearly true

We now check that

$P(n)$ implies $P(n+1)$.

if $P(n)$ is true, we have

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

and then

$$1+2+3+\cdots+n+(n+1) = \frac{n(n+1)}{2} + (n+1)$$

and so

$$1+2+\cdots+(n+1) = \frac{n(n+1)+2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

and so $P(n+1)$ would be true.

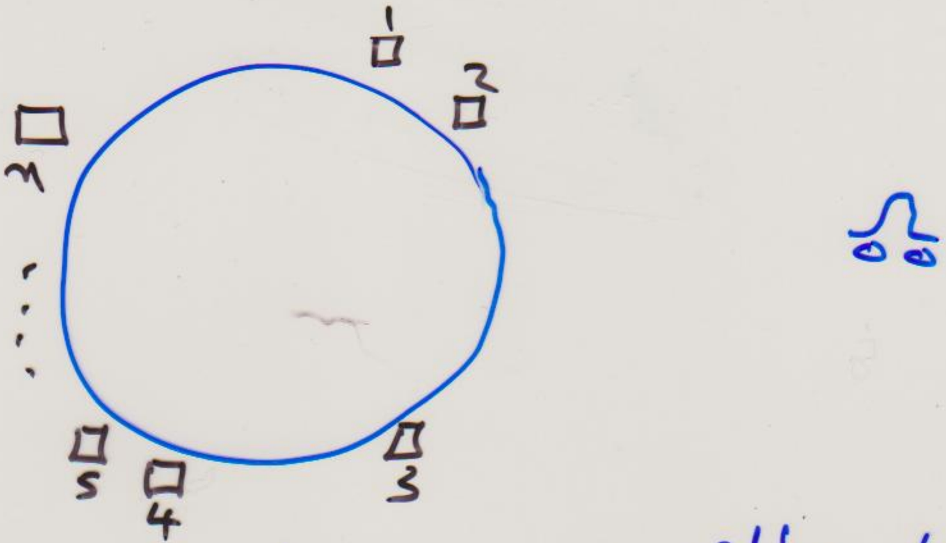
By the principle of induction the proposition $P(n)$ is true for all integers $n \geq 1$.

Principle of Induction v3

$$\left[P(1) \wedge \forall n, (P(n) \rightarrow P(n+1))\right] \rightarrow \forall n, P(n)$$

Example A car is on a circular track. There is enough petrol for the car to complete precisely one circuit. However, the petrol is distributed in n cans (different

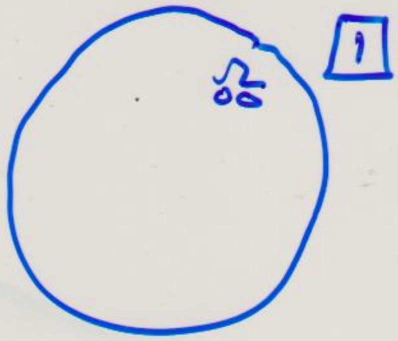
amounts in each can) and the cans are placed at random on the track.



Question: Is it always possible to choose a can at which to start, and then succeed in completing a circuit without running out of petrol?

P(n): It is always possible to complete a circuit by choosing an appropriate starting can.

P(1)

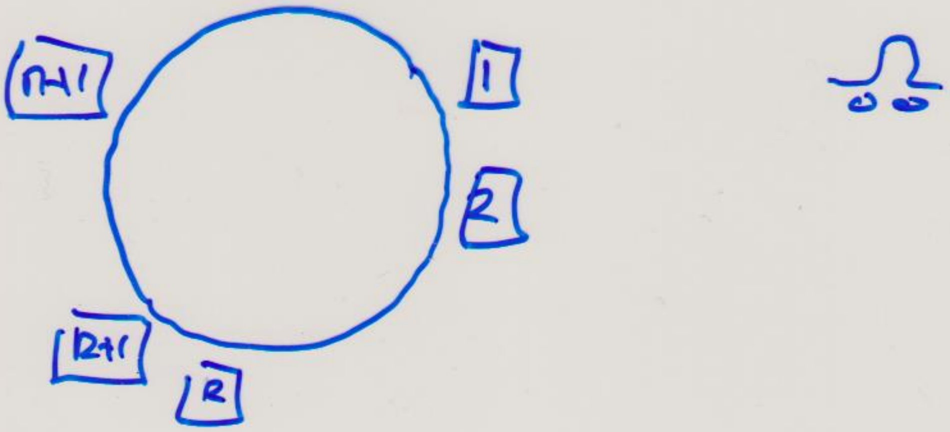


clearly true.

Let's try to prove

$$P(n) \longrightarrow P(n+1).$$

Suppose we have $n+1$ cans and that $P(n)$ is true.



There must be some can, can k, with enough petrol to get from it to can $k+1$. So we might as well pour the petrol from can $k+1$ into can k. The problem of $n+1$ cans is thus reduced to the problem of n cans, which

we can solve,

So $P(n) \rightarrow P(n+1)$.

By induction $P(n)$ is true for all $n \geq 1$.