Consider

$f(x) = x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1$

in $\mathbb{Z}_3[x]$.

<u>Definition</u> For $a \in \mathbb{Z}_3$ we say that a is a <u>root</u> of $f(x)$ if $(x-a)$ divides $f(x)$ with no remainder.

<u>Problem 1</u> Show that 1 is a root of $f(x) \in \mathbb{Z}_3[x]$ above.

<u>Sol^n</u>

$f(1) \equiv 0 \mod 3$.

so $(x-1)$ divides $f(x)$, so 1 is a root of $f(x)$.

<u>Defn</u> We say that a root a of $f(x)$ has <u>multiplicity</u> m if $(x-a)^m$ divides $f(x)$ with no remainder but $(x-a)^{m+1}$ does not.

Problem 2 For $f(x) \in \mathbb{Z}_3[x]$ above find the multiplicity of the root 1.

Sol^n 2 We could just work out

$$\frac{f(x)}{(x-1)}, \quad \frac{f(x)}{(x-1)^2}, \ldots$$

until we find the highest power of $(x-1)$ dividing $f(x)$.

Instead, let's think!

if $f(x) = (x-1)^m q(x)$ $(m \geq 1)$

then the "derivative" is

$$f'(x) = m(x-1)^{m-1} q(x) + (x-1)^m q'(x).$$

So

$$f'(x) = (x-1)^{m-1}\left[m\, q(x) + (x-1)\, q'(x)\right]$$

and $(x-1)^{m-1}$ divides $f'(x)$ with no remainder.

$\boxed{\text{Fact: } a \text{ is a root of } f(x) \text{ of multiplicity } \geq m \text{ if } a \text{ is a root of } f'(x) \text{ of multiplicity } m-1.}$

consider

$f(x) = x^6 + 2x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1$

in $\mathbb{Z}_3[x]$.

we know

$f(1) \equiv 0 \mod 3$

and that 1 is a root of $f(x)$.

$f'(x) = x^4 + 2x^3 + x + 2$

$f'(1) \equiv 0 \mod 3$

now consider

$f''(x) = x^3 + 1$

$f''(1) \not\equiv 0 \mod 3.$

Conclusion :

1 is not a root of $f''(x)$

1 is a root of $f'(x)$ of multiplicity at least 1

1 is a root of $f(x)$ of multiplicity ≥ 2.

i.e. $(x-1)^2$ divides $f(x)$ with no remainder.

<u>Problem</u> show that 1 is a root of

$$g(x) = x^6 - x^5 - x^4 + 2x^3 - x^2 - x + 1$$

in $\mathbb{Q}[x]$.

<u>Sol^n</u>

$g(1) = 0$. Hence 1 is a root of $g(x)$.

$$g'(x) = 6x^5 - 5x^4 - 4x^3 + 6x^2 - 2x - 1$$

$g'(1) = 0$. Hence 1 is a root of $g'(x)$.

$g''(x) = 30x^4 - 20x^3 - 12x^2 + 12x - 2$

$g''(1) \neq 0$.

Conclusion: 1 is a root of $g(x)$ of multiplicity ≥ 2.

Fact: For polynomials over $\mathbb{Q}$ this method gives the precise multiplicity.