Recall A random permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 4 & 5 & 8 & 7 & 6 & 9 & 2 & 11 & 12 & 1 & 10 \end{pmatrix}$$

can be written as a product of disjoint cycles

$$\pi = (1\ 3\ 5\ 7\ 9\ 11)(2\ 4\ 8)(10\ 12)$$

The lengths of the disjoint cycles (in this case 6, 3, 2) are uniquely determined by π.

We can also express π (not uniquely) as a product of transpositions.

$$\pi = (1\ 11)(1\ 9)(1\ 7)(1\ 5)(1\ 3)(2\ 8)(2\ 4)(10\ 12).$$

We say that the permutation π is <u>even</u> because it is a product of an even number of transpositions.

Defn The order of a permutation is the least positive integer n such that

$$\underbrace{\pi\,\pi\,\pi\cdots\pi}_{n \text{ times}} = (\ )$$

Example Consider

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 5 & 1 \end{pmatrix}$$

$$\begin{array}{ccccccccccc}
1 & & 3 & & 2 & & 4 & & 5 & & 1 \\
2 & & 4 & & 5 & & 1 & & 3 & & 2 \\
3 & \xrightarrow{\pi} & 2 & \xrightarrow{\pi} & 4 & \xrightarrow{\pi} & 5 & \xrightarrow{\pi} & 1 & \xrightarrow{\pi} & 3 \\
4 & & 5 & & 1 & & 3 & & 2 & & 4 \\
5 & & 1 & & 3 & & 2 & & 4 & & 5
\end{array}$$

since $\pi^5 = (\ )$ we say π has order 5.

FACT The order of any cycle

$$\pi = (1\ 3\ 4\ 2\ 6\ 5)$$

is equal to its length.

check $\pi^6 = (\ )$.

FACT The order of a product of disjoint cycles is equal to the lowest common multiple of the lengths of the cycles.

Example The order of

$$\pi = (1\ 3\ 5\ 7\ 9\ 11)(2\ 4\ 8)(10\ 11)$$

is

$$\mathrm{lcm}(6, 3, 2) = 6.$$

Algorithm: to find order of a permutation.

Input: a permutation π

Output: the order of π

procedure: first express π as a product of disjoint cycles. Then output the lcm of the lengths of the cycles.

The 15 - puzzle

Suppose given a 4×4 puzzle of sliding squares, with one blank, in the initial state

1	15	14	13
12	11	10	9
8	7	6	5
4	3	2	

Suppose we want to slide squares to achieve the state

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Let's pretend that the blank square is numbered 16.

Let's also imagine that the sixteen squares are lying on a red/white chequered board.

In our language, each slide is a transposition, and we want to find a sequence of transpositions that compose to give the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 15 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 16 \end{pmatrix}$$

This π is

$$(2\ 15)(3\ 14)(4\ 13)(5\ 12)(6\ 11)$$

$$\ldots (7\ 10)(8\ 9)$$

So π is odd. But an odd number of transpositions would leave the blank on a red background. So it is not possible to solve the puzzle from initial state.