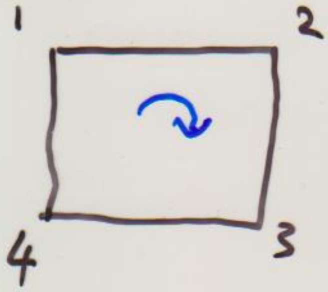
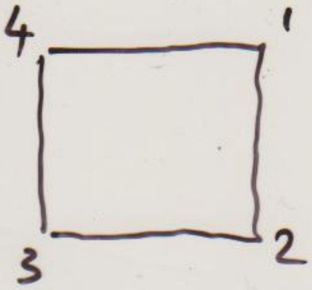
Symmetries of a square

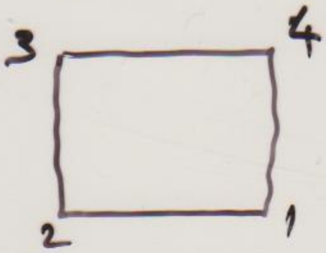


rotate through $\frac{\pi}{2}$



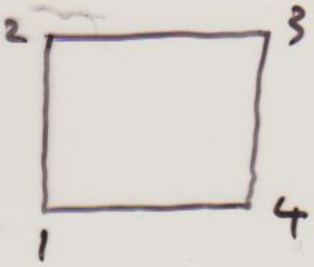
(1 4 3 2)

rotate by π



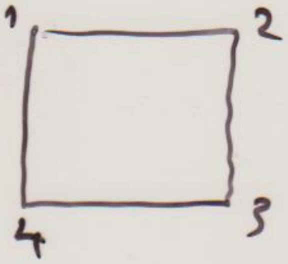
(1 3) (2 4)

rotate by $3\pi/2$



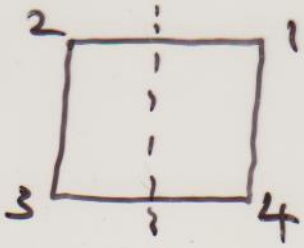
(1 2 3 4)

identity



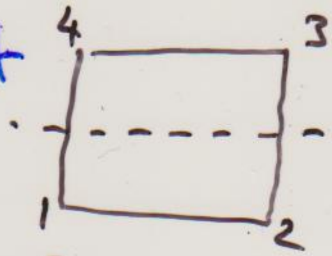
( )

reflect



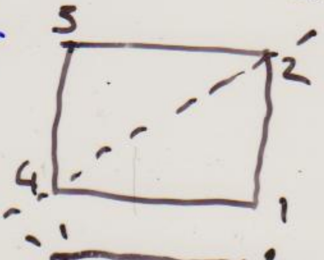
(1 2) (3 4)

reflect



(1 4) (2 3)

reflect



(1 3)

(2 4)

Defn A cycle is a permutation of the form (3 4) or (3 1 2 4), or (1 2 3 4 5).

Note: In the symmetries of a square every permutation is either a cycle or a product of cycles.

FACT: Any permutation can be written as a product of disjoint cycles. (Two cycles are disjoint if they involve no common numbers.)

Example

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 9 & 3 & 7 & 2 & 6 & 10 & 8 & 5 & 1 \end{pmatrix}$$

$$= (1\ 4\ 7\ 10)(2\ 9\ 5)$$

Definition A cycle of length 2 is called a transposition.

FACT: Any cycle can be written (in many different ways) as a product of transposition. (The transpositions might not be disjoint.)

Example

$(2\ 9\ 5) = (2\ 5)(2\ 9)$

$(1\ 4\ 7\ 10) = (1\ 10)(1\ 7)(1\ 4)$

alternatively

$(1\ 4\ 7\ 10) = (4\ 7\ 10\ 1) = (7\ 4)(7\ 10)(4\ 7)$

We've proved:

Theorem Any permutation can be written as a product of transpositions.

Example

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 9 & 3 & 7 & 2 & 6 & 10 & 8 & 5 & 1 \end{pmatrix}$$

$$= (1\ 4\ 7\ 10)(2\ 9\ 5)$$

$$= (1\ 10)(1\ 7)(1\ 4)(2\ 5)(2\ 9)$$

Also

$$\pi = (8\ 3)(7\ 4)(7\ 10)(4\ 7)(2\ 5)(2\ 9)(3\ 8)$$

FACT: Any permutation π can

either:
only be written as a product of an even number of transpositions,

or:
only be written as a product of an odd number of transpositions.

Defn A permutation is odd if it is a product of an odd number of transpositions, Otherwise it is even.

Symmetries of a square

$(1\ 4\ 3\ 2) = (1\,2)(1\,3)(1\,4)$ odd

$(1\ 3)(2\ 4)$ even

$(\ )$ even

$(1\ 2)(3\ 4)$ even

$(1\ 4)(2\ 3)$ even

$(1\ 2)$ odd

$(2\ 4)$ odd

$(1\ 2\ 3\ 4) = (1\,4)(1\,3)(1\,2)$ odd

Four even permutations and four odd permutations.