Permutations

Let X be a finite set.
A function $f : X \to X$ is called a **permutation** if it is injective and surjective.

Example $X = \{1, 2, 3\}$

$f_1 : X \to X$, $1 \mapsto 1$, $2 \mapsto 2$, $3 \mapsto 3$ or $f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$f_2 : X \to X$ $1 \mapsto 2$, $2 \mapsto 1$, $3 \mapsto 3$ $f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$f_3 : X \to X$ $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$ $f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$f_4 : X \to X$ $1 \mapsto 3$, $2 \mapsto 2$, $3 \mapsto 1$ $f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$f_5 : X \to X$ $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 2$ $f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$f_6 : X \to X$ $1 \mapsto 1$, $2 \mapsto 3$, $3 \mapsto 2$ $f_6 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

We can compose permutations to get new permutations.

$f_2 \circ f_3 : X \to X$, $1 \mapsto f_2(f_3(1)) = 1$

$2 \mapsto f_2(f_3(2)) = 3$

$3 \mapsto \qquad = 2$

$f_2 \circ f_3 = f_6$

$f_3 \circ f_2 : X \to X$ $\quad 1 \longmapsto 3$

$2 \longmapsto 2$

$3 \longmapsto 1$

$f_3 \circ f_2 = f_4$

So $f_3 \circ f_2 \neq f_2 \circ f_3$

Alternative notation for permutations

$f_1 = (\ )$

$f_2 = (1\ 2)$

$f_3 = (1\ 2\ 3)$

$f_4 = (1\ 3)$

$f_5 = (1\ 3\ 2)$

$f_6 = (2\ 3)$

Observation

$$f_1 \circ f_k = f_k \qquad (k = 1, 2, \cdots, 6)$$

$$f_k \circ f_1 = f_k$$

We call $f_1 = (\ )$ the identity permutation.

Definition The inverse of a permutation f is a permutation f^{-1} such that

$$f f^{-1} = (\ )$$

$$f^{-1} f = (\ ).$$

Example

$f_3 = (1\ 2\ 3)$

$f_3^{-1} = (3\ 2\ 1)$

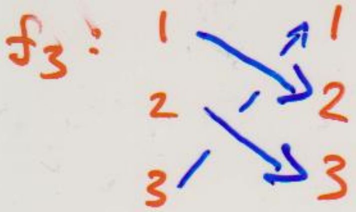
since

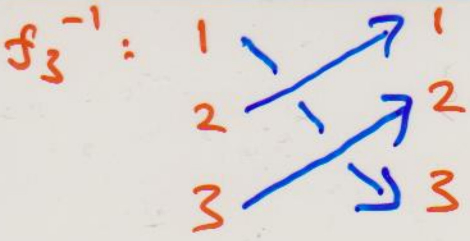
$$(1\ 2\ 3)(3\ 2\ 1) = (\ )$$

$$(3\ 2\ 1)(1\ 2\ 3) = (\ )$$

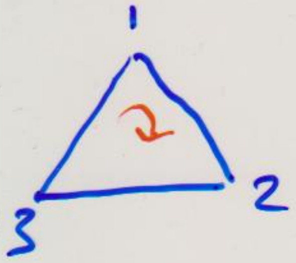
$f_4 = (1\ 3)$ since $(1\ 3)(1\ 3) = (\ )$

$f_4^{-1} = (1\ 3)$

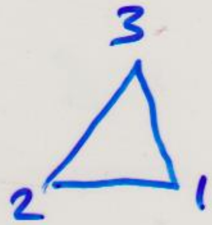




Permutations arise from geometry

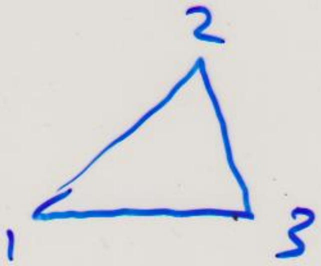


rotate through $\frac{2\pi}{3}$



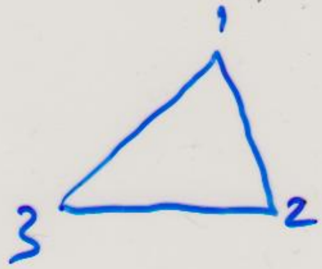
(1 3 2)

rotate through $\frac{4\pi}{3}$



(1 2 3)

rotate through 0



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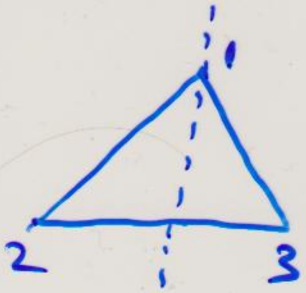
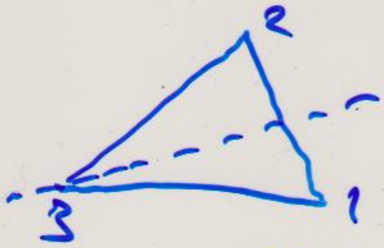
reflect



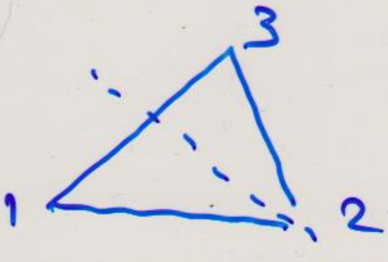
(2 3)

reflect



(1 2)

reflect



(1 3)