Example $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$

xRy means "$x-y$ is an integer multiple of 3"

From yesterday, we know that this is an equivalence relation X.

$[0] = \{0, 3, 6\}$

$[1] = \{1, 4, 7\}$

$[2] = \{2, 5\}$

Note:

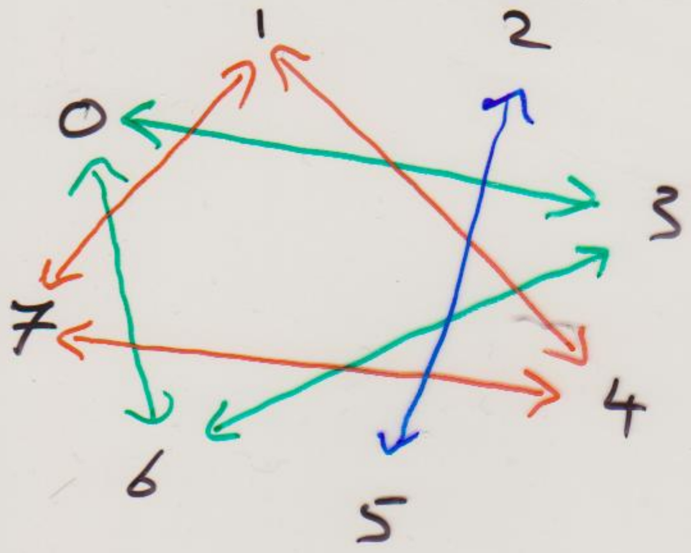
$X = [0] \cup [1] \cup [2]$

$[0] \cap [1] = \emptyset$ (empty set)

$[0] \cap [2] = \emptyset$

$[1] \cap [2] = \emptyset$

We can represent this relation R by an "arrow diagram".



<u>Definition</u> Let X be a set, and let $A, B, C, \ldots$ be subsets of X. We say that these subsets form a <u>partition</u> of X if

$$X = A \cup B \cup C \cup \ldots$$

and the intersection of any two distinct subsets is empty.

Example $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$A = \{0, 3, 6\}$

$B = \{1, 4, 7\}$

$C = \{2, 5\}$.

This is a partition of X.

Theorem If $A, B, C, \ldots$ is a partition of X, then we can define an equivalence relation R on X by

xRy means "x and y are elements of a common subset in the partition"

Problem Consider the relation on $\mathbb{Z}$ given by

$$R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x^2 - y^2 \text{ is is an integer multiple of } 5\}$$

show that this is an equivalence relation, and describe the corresponding partition of $\mathbb{Z}$.

Sol^n

$R = \{(4,1), (-4,1), (4,-1), (0,0), \ldots$

check reflexivity:
For any $x \in \mathbb{Z}$ we have

$$x R x$$

since $x^2 - x^2 = 0$ is an integer multiple of 5. ✓

check symmetric property:
For any $x, y \in \mathbb{Z}$
if $x R y$ then $y R x$
since $x^2 - y^2 = 5k$ implies

$y^2 - x^2 = 5(-k)$.

Check transitivity!

If xRy and yRz

then

$x^2 - y^2 = 5k$ and $y^2 - z^2 = 5m$

and so

$$x^2 - z^2 = (x^2 - y^2) + (y^2 - z^2)$$
$$= 5k + 5m$$
$$= 5(k+m)$$

and xRz.

To describe the partition of $\mathbb{Z}$ let's think.

xRy means $x^2 - y^2$ is a multiple of 5

means $x^2 \equiv y^2 \mod 5$

working mod 5 :

$0^2 = 0$

$1^2 = 1$

$2^2 = 4$

$3^2 = 4$

$4^2 = 1$

so the partition of $\mathbb{Z}$ corresponding to R is

$[0] = \{5k : k \in \mathbb{Z}\}$

$[1] = \{1+5k : k \in \mathbb{Z}\} \cup \{4+5k : k \in \mathbb{Z}\}$

$[2] = \{2+5k : k \in \mathbb{Z}\} \cup \{3+5k : k \in \mathbb{Z}\}$